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A HISTORY OF ELEMENTARY MATHEMATICS

BOOKS BY FLORIAN CAJORI

History of Mathematics
Revised and Enlarged Edition

History of Elementary Mathematics
Revised and Enlarged Edition

History of Physics

Introduction to the Modern
Theory of Equations

A HISTORY

OF

ELEMENTARY MATHEMATICS

WITH

HINTS ON METHODS OF TEACHING

BY

FLORIAN CAJORI, PH.D.

PROFESSOR OF THE HISTORY OF MATHEMATICS IN THE UNIVERSITY OF CALIFORNIA

REVISED AND ENLARGED EDITION

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PREFACE TO THE FIRST EDITION

"The education of the child must accord both in mode and arrangement with the education of mankind as considered historically; or, in other words, the genesis of knowledge in the individual must follow the same course as the genesis of knowledge in the race. To M. Comte we believe society owes the enunciation of this doctrine — a doctrine which we may accept without committing ourselves to his theory of the genesis of knowledge, either in its causes or its order."[1] If this principle, held also by Pestalozzi and Froebel, be correct, then it would seem as if the knowledge of the history of a science must be an effectual aid in teaching that science. Be this doctrine true or false, certainly the experience of many instructors establishes the importance of mathematical history in teaching.[2] With the hope of being of some assistance to my fellow-teachers, I have prepared this book and have interlined my narrative with occasional remarks and suggestions on methods of teaching. No doubt, the thoughtful reader will draw many useful

[1] Herbert Spencer, *Education: Intellectual, Moral, and Physical.* New York, 1894, p. 122. See also R. H. Quick, *Educational Reformers*, 1879, p. 191.

[2] See G. Heppel, "The Use of History in Teaching Mathematics," *Nature*, Vol. 48, 1893, pp. 16–18.

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lessons from the study of mathematical history which are not directly pointed out in the text.

In the preparation of this history, I have made extensive use of the works of Cantor, Hankel, Unger, De Morgan, Peacock, Gow, Allman, Loria, and of other prominent writers on the history of mathematics. Original sources have been consulted, whenever opportunity has presented itself. It gives me much pleasure to acknowledge the assistance rendered by the United States Bureau of Education, in forwarding for examination a number of old text-books which otherwise would have been inaccessible to me. It should also be said that a large number of passages in this book are taken, with only slight alteration, from my *History of Mathematics*, Macmillan & Co., 1895. . . .

FLORIAN CAJORI.

COLORADO COLLEGE, COLORADO SPRINGS,
July, 1896.

PREFACE TO THE SECOND EDITION

IN the endeavor to bring this history down to date, numerous alterations and additions have been made.

FLORIAN CAJORI.

COLORADO COLLEGE,
December, 1916.

CONTENTS

PAGE

ANTIQUITY 1

NUMBER-SYSTEMS AND NUMERALS 1

ARITHMETIC AND ALGEBRA 19

Egypt 19

Greece 26

Rome 37

GEOMETRY AND TRIGONOMETRY 43

Egypt and Babylonia 43

Greece 46

Rome 89

MIDDLE AGES 93

ARITHMETIC AND ALGEBRA 93

Hindus 93

Arabs 103

Europe during the Middle Ages 111

Introduction of Roman Arithmetic 111

Translation of Arabic Manuscripts 118

The First Awakening 119

GEOMETRY AND TRIGONOMETRY 122

Hindus 122

Arabs 125

Europe during the Middle Ages 131

Introduction of Roman Geometry 131

Translation of Arabic Manuscripts 132

The First Awakening 134

PAGE

MODERN TIMES 139

ARITHMETIC 139

Its Development as a Science and Art 139

English Weights and Measures 167

Rise of the Commercial School in England 179

Causes which Checked the Growth of Demonstrative Arithmetic in England 204

Reforms in Arithmetical Teaching 211

Arithmetic in the United States 215

"*Pleasant and Diverting Questions*" 219

ALGEBRA 224

The Renaissance 224

The Last Three Centuries 234

GEOMETRY AND TRIGONOMETRY 245

Editions of Euclid. Early Researches 245

The Beginning of Modern Synthetic Geometry . . . 252

Modern Elementary Geometry 256

Modern Synthetic Geometry 257

Modern Geometry of the Triangle and Circle . . 259

Non-Euclidean Geometry 266

Text-books on Elementary Geometry 275

RECENT MOVEMENTS IN TEACHING 290

The Perry Movement 291

International Commission 297

American Associations 301

Attacks upon the Study of Mathematics as a Training of the Mind 304

A HISTORY OF MATHEMATICS

ANTIQUITY

NUMBER-SYSTEMS AND NUMERALS

Nearly all number-systems, both ancient and modern, are based on the scale of 5, 10, or 20. The reason for this it is not difficult to see. When a child learns to count, he makes use of his fingers and perhaps of his toes. In the same way the savages of prehistoric times unquestionably counted on their fingers and in some cases also on their toes. Such is indeed the practice of the African, the Eskimo, and the South Sea Islander of to-day.[1] This recourse to the fingers has often resulted in the development of a more or less extended pantomime number-system, in which the fingers were used as in a deaf and dumb alphabet.[1] Evidence of the prevalence of finger symbolisms is found among the ancient Egyptians, Babylonians, Greeks, and Romans, as also among the Europeans of the middle ages: even now nearly all Eastern nations use finger symbolisms. The Chinese express on the left hand

[1] L. L. Conant, "Primitive Number-Systems," in *Smithsonian Report*, 1892, p. 584.

"all numbers less than 100,000; the thumb nail of the right hand touches each joint of the little finger, passing first up the external side, then down the middle, and afterwards up the other side of it, in order to express the nine digits; the tens are denoted in the same way, on the second finger; the hundreds on the third; the thousands on the fourth; and ten-thousands on the thumb. It would be merely necessary to proceed to the right hand in order to be able to extend this system of numeration."[1] So common is the use of this finger-symbolism that traders are said to communicate to one another the price at which they are willing to buy or sell by touching hands, the act being concealed by their cloaks from observation of by-standers.

Had the number of fingers and toes been different in man, then the prevalent number-systems of the world would have been different also. We are safe in saying that had one more finger sprouted from each human hand, making twelve fingers in all, then the numerical scale adopted by civilized nations would not be the decimal, but the duodecimal. Two more symbols would be necessary to represent 10 and 11, respectively. As far as arithmetic is concerned, it is certainly to be regretted that a sixth finger did not appear. Except for the necessity of using two more signs or numerals and of being obliged to learn the multiplication table as far as 12×12, the duodecimal system is decidedly superior to the decimal. The number twelve has for its exact divisors 2, 3, 4, 6, while ten has only 2 and 5. In ordinary business affairs, the fractions $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, are used extensively, and it is very convenient to have a base which is an exact multiple of 2, 3, and 4. Among the most zealous advocates of the duodecimal scale was Charles XII.

[1] George Peacock, article "Arithmetic," in *Encyclopædia Metropolitana* (*The Encyclopædia of Pure Mathematics*), p. 394. Hereafter we shall cite this very valuable article as Peacock.

of Sweden, who, at the time of his death, was contemplating the change for his dominions from the decimal to the duodecimal.[1] But it is not likely that the change will ever be brought about. So deeply rooted is the decimal system that when the storm of the French Revolution swept out of existence other old institutions, the decimal system not only remained unshaken, but was more firmly established than ever. The advantages of twelve as a base were not recognized until arithmetic was so far developed as to make a change impossible. "The case is the not uncommon one of high civilization bearing evident traces of the rudeness of its origin in ancient barbaric life."[2]

Of the notations based on human anatomy, the quinary and vigesimal systems are frequent among the lower races, while the higher nations have usually avoided the one as too scanty and the other as too cumbrous, preferring the intermediate decimal system.[3] Peoples have not always consistently adhered to any one scale. In the quinary system, 5, 25, 125, 625, etc., should be the values of the successive higher units, but a quinary system thus carried out was never in actual use: whenever it was extended to higher numbers it invariably ran either into the decimal or into the vigesimal system. "The home par excellence of the quinary, or rather of the quinary-vigesimal scale, is America. It is practically universal among the Eskimo tribes of the Arctic regions. It prevailed among a considerable portion of the North American Indian tribes, and was almost universal with the native races of Central and

[1] Conant, *op. cit.*, p. 589.

[2] E. B. Tylor, *Primitive Culture*, New York, 1889, Vol. I., p. 272. In some respects a scale having for its base a power of 2 — the base 8 or 16, for instance, — is superior to the duodecimal, but it has the disadvantage of not being divisible by 3. See W. W. Johnson, "Octonary Numeration," *Bull. N. Y. Math. Soc.*, 1891, Vol. I., pp. 1–6.

[3] Tylor, *op. cit.*, Vol. I., p. 262.

South America."[1] This system was used also by many of the North Siberian and African tribes. Traces of it are found in the languages of peoples who now use the decimal scale; for example, in Homeric Greek. The Roman notation reveals traces of it; viz., I, II, ··· V, VI, ··· X, XI, ··· XV, etc.

It is curious that the quinary should so frequently merge into the vigesimal scale; that savages should have passed from the number of fingers on one hand as an upper unit or a stopping-place, to the total number of fingers and toes as an upper unit or resting-point. The vigesimal system is less common than the quinary, but, like it, is never found entirely pure. In this the first four units are 20, 400, 8000, 160,000, and special words for these numbers are actually found among the Mayas of Yucatan. The transition from quinary to vigesimal is shown in the Aztec system, which may be represented thus, $1, 2, 3, 4, 5, 5+1, \cdots 10, 10+1, \cdots 10+5, 10+5+1, \cdots 20, 20+1, \cdots 20+10, 20+10+1, \cdots 40$, etc.[2] Special words occur here for the numbers 1, 2, 3, 4, 5, 10, 20, 40, etc. The vigesimal system flourished in America, but was rare in the Old World. Celtic remnants of one occur in the French words *quatre-vingts* (4×20 or 80), *six-vingts* (6×20 or 120), *quinze-vingts* (15×20 or 300). Note also the English word *score* in such expressions as *three-score years and ten.*

Of the three systems based on human anatomy, the decimal system is the most prevalent, so prevalent, in fact, that according to ancient tradition it was used by all the races of the world. It is only within the last few centuries that the other

[1] Conant, *op. cit.*, p. 592. For further information see also Pott, *Die quinäre und vigesimale Zählmethode bei Völkern aller Welttheile*, Halle, 1847; Pott, *Die Sprachverschiedenheit in Europa an den Zahlwörtern nachgewiesen, sowie die quinäre und vigesimale Zählmethode*, Halle, 1868.

[2] Tylor, *op. cit.*, Vol. I., p. 262.

two systems have been found in use among previously unknown tribes.[1] The decimal scale was used in North America by the greater number of Indian tribes, but in South America it was rare.

In the construction of the decimal system, 10 was suggested by the number of fingers as the first stopping-place in counting, and as the first higher unit. Any number between 10 and 100 was pronounced according to the plan $b(10) + a(1)$, a and b being integers less than 10. But the number 110 might be expressed in two ways, (1) as $10 \times 10 + 10$, (2) as 11×10. The latter method would not seem unnatural. Why not imitate *eighty*, *ninety*, and say *eleventy*, instead of *hundred and ten ?* But upon this choice between $10 \times 10 + 10$ and 11×10 hinges the systematic construction of the number system.[2] Good luck led all nations which developed the decimal system to the choice of the former;[3] the unit 10 being here treated in a manner similar to the treatment of the lower unit 1 in expressing numbers below 100. Any number between 100 and 1000 was designated $c(10)^2 + b(10) + a$, a, b, c representing integers less than 10. Similarly for numbers below 10,000, $d(10)^3 + c(10)^2 + b(10)^1 + a(10)^0$; and similarly for still higher numbers.

Proceeding to describe the *notations* of numbers, we begin with the Babylonian. Cuneiform writing, as also the accompanying notation of numbers, was probably invented

[1] Conant, *op. cit.*, p. 588.

[2] Hermann Hankel, *Zur Geschichte der Mathematik in Alterthum und Mittelalter*, Leipzig, 1874, p. 11. Hereafter we shall cite this brilliant work as Hankel.

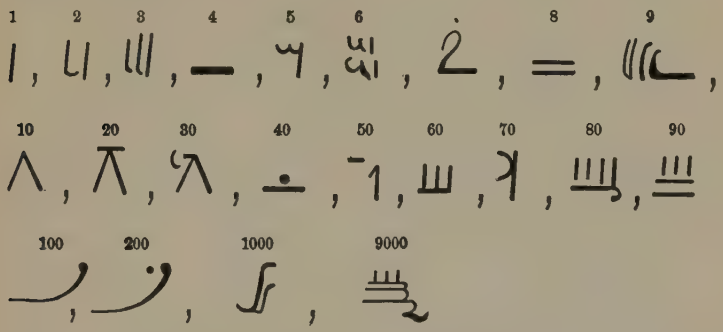
[3] In this connection read also Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, Vol. I. (Third Edition), Leipzig, 1907, pp. 6 and 7. This history, by the prince of mathematical historians of the nineteenth century, is in four volumes, and will be cited hereafter as Cantor.

by the early Sumerians. A vertical wedge stood for one, while and signified 10 and 100, respectively. In case of numbers below 100, the values of the separate symbols were *added*. Thus, for 23, for 30. The signs of higher value are written on the left of those of lower value. But in writing the hundreds a smaller symbol was placed before that for 100 and was *multiplied* into 100. Thus, signified 10×100 or 1000. Taking this for a new unit, was interpreted, not as 20×100, but as 10×1000. In this notation no numbers have been found as large as a million.[1] The principles applied in this notation are the *additive* and the *multiplicative*. Besides this the Babylonians had another, the sexagesimal notation, to be noticed later.

An insight into Egyptian methods of notation was obtained through the deciphering of the hieroglyphics by Champollion, Young, and others. The numerals are (1), (10), (100), (1000), (10,000), (100,000), (1,000,000), (10,000,000). The sign for one represents a vertical staff; that for 10,000, a pointing finger; that for 100,000, a burbot; that for 1,000,000, a man in astonishment. No certainty has been reached regarding the significance of the other symbols. These numerals like the other hieroglyphic signs were plainly pictures of animals or objects familiar to the Egyptians, which in some way suggested the idea to be conveyed. They are excellent examples of picture-writing. The principle involved in the Egyptian notation was the *additive* throughout. Thus, would be 111.

[1] For fuller treatment see Moritz Cantor, *Mathematische Beiträge zum Kulturleben der Völker*, Halle, 1863, pp. 22–38.

Hieroglyphics are found on monuments, obelisks, and walls of temples. Besides these the Egyptians had *hieratic* and *demotic* writings, both supposed to be degenerated forms of hieroglyphics, such as would be likely to evolve through prolonged use and attempts at rapid writing. The following are hieratic signs:[1]



Since there are more hieratic symbols than hieroglyphic, numbers could be written more concisely in the former. The additive principle rules in both, and the symbols for larger values always precede those for smaller values.

About the time of Solon, the Greeks used the initial letters of the numeral adjectives to represent numbers. These signs are often called *Herodianic signs* (after Herodianus, a Byzantine grammarian of about 200 A.D., who describes them). They are also called *Attic*, because they occur frequently in Athenian inscriptions. The Phœnicians, Syrians, and Hebrews possessed at this time alphabets and the two latter used letters of the alphabet to designate numbers. The Greeks began to adopt the same course about 500 B.C. The letters of the Greek alphabet, together with three antique letters, ϛ, ϙ, Ϡ, and the

[1] CANTOR, Vol. I., pp. 84 and 85. The hieratic numerals are taken from Cantor's table at the end of the volume published in 1894.

symbol M, were used for numbers. For the numbers 1–9 they wrote α, β, γ, δ, ϵ, ς, ζ, η, θ; for the tens 10–90, ι, κ, λ, μ, ν, ξ, o, π, φ, for the hundreds 100–900, ρ, σ, τ, υ, ϕ, χ, ψ, ω, $\mathfrak{D}$; for the thousands they wrote $_{,}\alpha$, $_{,}\beta$, $_{,}\gamma$, $_{,}\delta$, $_{,}\epsilon$, etc.; for 10,000, M; for 20,000, $\overset{\beta}{\mathrm{M}}$; for 30,000, $\overset{\gamma}{\mathrm{M}}$, etc. The change from Attic to alphabetic numerals was decidedly for the worse, as the former were less burdensome to the memory. In Greek grammars we often find it stated that alphabetic numerals were marked with an accent to distinguish them from words, but this was not commonly the case; a horizontal line drawn over the number usually answered this purpose, while the accent generally indicated a unit-fraction, thus $\delta' = \frac{1}{4}$.[1] The Greeks applied to their numerals the additive and, in cases like $\overset{\epsilon}{\mathrm{M}}$ for 50,000, also the multiplicative principle.

In the Roman notation we have, besides the additive, the principle of subtraction. If a letter is placed before another of greater value, the former is to be subtracted from the latter. Thus, IV = 4, while VI = 6. Though this principle has not been found in any other *notation*, it sometimes occurs in *numeration*. Thus in Latin *duodeviginti* = 2 from 20, or 18.[2] The Roman numerals are supposed to be of Etruscan origin.

Thus, in the Babylonian, Egyptian, Greek, Roman, and other decimal notations of antiquity, numbers are expressed by means of a few signs, these symbols being combined by addition alone, or by addition together with multiplication or subtraction. But in none of these decimal systems do we find the all-important principle of position or principle of

[1] Dr. G. Friedlein, *Die Zahlzeichen und das Elementare Rechnen der Griechen und Römer*, Erlangen, 1869, p. 13. The work will be cited after this as Friedlein. See also Dr. Siegmund Günther in Müller's *Handbuch der Klassischen Altertumswissenschaft*, Fünfter Band, 1. Abteilung, 1888, p. 9.

[2] Cantor, Vol. I., pp. 11 and 525.

local value, such as we have in the notation now in use. Having missed this principle, the ancients had no use for a symbol to represent zero, and were indeed very far removed from an ideal notation. In this matter even the Greeks and Romans failed to achieve what a remote nation in Asia, little known to Europeans before the present century, accomplished most admirably. But before we speak of the Hindus, we must speak of an ancient Babylonian notation, which, strange to say, is not based on the scale 5, 10, or 20, and which, moreover, came very near a full embodiment of the ideal principle found wanting in other ancient systems. We refer to the *sexagesimal* notation.

The Babylonians used this chiefly in the construction of weights and measures. The systematic development of the sexagesimal scale, both for integers and fractions, reveals a high degree of mathematical insight on the part of the early Sumerians. The notation has been found on two Babylonian tablets. One of them, probably dating from 1600 or 2300 B.C., contains a list of square numbers up to 60^2. The first seven are 1, 4, 9, 16, 25, 36, 49. We have next $1.4 = 8^2$, $1.21 = 9^2$, $1.40 = 10^2$, $2.1 = 11^2$, etc. This remains unintelligible, unless we assume the scale of sixty, which makes $1.4 = 60 + 4$, $1.21 = 60 + 21$, etc. The second tablet records the magnitude of the illuminated portion of the moon's disc for every day from new to full moon, the whole disc being assumed to consist of 240 parts. The illuminated parts during the first five days are the series 5, 10, 20, 40, $1.20 (= 80)$. This reveals again the sexagesimal scale and also some knowledge of geometrical progressions. From here on the series becomes an arithmetical progression, the numbers from the fifth to the fifteenth day being respectively, 1.20, 1.36, 1.52, 2.8, 2.24, 2.40, 2.56, 3.12, 3.28, 3.44, 4. In this sexagesimal notation we have, then, the principle of local value. Thus, in $1.4 (= 64)$, the 1 is

made to stand for 60, the unit of the second order, by virtue of its position with respect to the 4. In Babylonia some use was thus made of the principle of position, perhaps 2000 years before the Hindus developed it. This was at a time when Romulus and Remus, yea even Achilles, Menelaus, and Helen, were still unknown to history and song. But the full development of the principle of position calls for a symbol to represent the *absence* of quantity, or zero. Babylonian records of about 200 B.C. give a symbol for zero which denoted the absence of a figure, but apparently was not used in calculation. It consisted of two marks . About 130 A.D. Ptolemy in his *Almagest* used sexagesimal fractions, and also the omicron o to represent blanks in the sexagesimal numbers. This o was not used as a regular zero. Hence, the Babylonians had the principle of local value, and also a symbol for zero to indicate the absence of a figure, but did not use this zero in computation. Their sexagesimal fractions were transmitted to India.

What was it that suggested to the Babylonians the number sixty as a base? Cantor[1] and others offer the following provisional reply: At first the Babylonians reckoned the year at 360 days. This led to the division of the circumference of a circle into 360 degrees, each degree representing the daily part of the supposed yearly revolution of the sun around the earth. Probably they knew that the radius could be applied to the circumference as a chord six times, and that each arc thus cut off contained 60 degrees. Thus the division into 60 parts may have suggested itself. When greater precision was needed, each degree was divided into 60 equal parts, or minutes. In this way the sexagesimal notation may have originated. The division of the day into 24 hours, and of the hour into minutes and seconds on the scale of 60,

[1] Vol. I., pp. 91–93.

is due to the Babylonians. There are also indications of a knowledge of sexagesimal *fractions*,[1] such as were used later by the Greeks, Arabs, by scholars of the middle ages and of even recent times.

Babylonian science has made its impress upon modern civilization. Whenever a surveyor copies the readings from the graduated circle on his theodolite, whenever the modern man notes the time of day, he is, unconsciously perhaps, but unmistakably, doing homage to the ancient astronomers on the banks of the Euphrates.

The full development of our decimal notation belongs to comparatively modern times. Decimal notation had been in use for thousands of years, before it was perceived that its simplicity and usefulness could be enormously increased by the adoption of the principle of position. To the Hindus of the fifth or sixth century after Christ we owe the full and systematic development of the use of the zero and the principle of local value. Of all mathematical discoveries, no one has contributed more to the general progress of intelligence than this. While the older notations served merely to record the answer of an arithmetical computation, the Hindu notation (wrongly called the Arabic notation) assists with marvellous power in performing the computation itself. To verify this truth, try to multiply 723 by 364, by first expressing the numbers in the Roman notation; thus, multiply DCCXXIII by CCCLXIV. This notation offers little or no help; the Romans were compelled to invoke the aid of the abacus in calculations like this.

Very little is known concerning the mode of evolution of the Hindu notation. There is evidence for the belief that the Hindu notation of the second century, A.D., did not include

[1] CANTOR, Vol. I., p. 31.

the zero nor the principle of local value. On the island of Ceylon a notation resembling the Hindu, but without the zero, has been preserved. It is known that Buddhism and Indian culture were transplanted thither about the third century and there remained stationary. It is highly probable, then, that the notation of Ceylon is the old imperfect Hindu system. Besides signs for 1–9, the Ceylon notation has symbols for each of the tens and for 100 and 1000. Thus the number 7685 would have been written with six symbols, designating respectively the numbers 7, 1000, 6, 100, 80, 5. These so-called Singhalesian signs are supposed originally to have been, like the old Hindu numerals, the initial letters of the corresponding numeral adjectives.[1] Unlike the English, the first nine Sanscrit numeral adjectives have each a different beginning, thereby excluding ambiguity. In course of centuries the forms of the Hindu letters altered materially, but the letters that seem to resemble most closely the apices of Boethius and the West-Arabic numerals (which we shall encounter later) are the letters of the second century.

The earliest known form of the Indian symbol for zero was that of a dot, which was used in inscriptions and manuscripts in order to mark a blank. This early use resembles the still earlier use made of symbols for zero by the Babylonians and by Ptolemy. Probably Aryabhatta in the fifth century A.D. knew our zero. The earliest undoubted occurrence of our zero in India is in 876 A.D.

The Hindus possessed several different modes of designating numbers. They sometimes found it convenient to use a *symbolic system of position*, in which 1 might be expressed by "moon" or "earth," 2 by "eye," etc. In the *Surya-siddhanta*[2] (a text-

[1] CANTOR, Vol. I., pp. 603–607.

[2] Translated by E. BURGESS, and annotated by W. D. WHITNEY, New Haven, 1860, p. 3.

book of Hindu astronomy) the number 1577917828 is thus given: Vasu (a class of deities 8 in number) — two — eight — mountain (the 7 mythical chains of mountains) — form — figure (the 9 digits) — seven — mountain — lunar days (of which there are 15 in the half-month). This notation is certainly interesting. It seems to have been applied as a *memoria technica* in order to record dates and numbers. Such a selection of synonyms made it much easier to draw up phrases or obscure verses for artificial memory. To a limited degree this idea may perhaps be advantageously applied by the teacher in the schoolroom.

The Hindu notation, in its developed form, reached Europe during the twelfth century. It was transmitted to the Occident through the Arabs, hence the name "Arabic notation." No blame attaches to the Arabs for this pseudo-name; they always acknowledged the notation as an inheritance from India. During the 1000 years preceding 1200 A.D., the Hindu numerals and notation, while in the various stages of evolution, were carried from country to country. Exactly what these migrations were, is a problem of extreme difficulty. Not even the authorship of the letters of Junius has produced so much discussion.[1] The facts to be explained and harmonized are as follows:

1. When, toward the close of the last century, scholars gradually became convinced that our numerals were not of Arabic, but of Hindu origin, the belief was widespread that the Arabic and Hindu numerals were essentially identical in form. Great was the surprise when a set of Arabic numerals, the so-called *Gubar-numerals*, was discovered, some of which

[1] Consult TREUTLEIN, *Geschichte unserer Zahlzeichen und Entwickelung der Ansichten über dieselbe*, Karlsruhe, 1875; SIEGMUND GÜNTHER, *Ziele und Resultate der neueren mathematisch-historischen Forschung*, Erlangen, 1876, note 17.

bore *no resemblance whatever* to the modern Hindu characters, called *Devanagari-numerals.*

2. Closer research showed that the numerals of the Arabs of Bagdad *differed* from those of the Arabs at Cordova, and this to such an extent that it was difficult to believe the westerners received the digits directly from their eastern neighbours. The West-Arabic symbols were the Gubar-numerals mentioned above. The Arabic digits can be traced back to the tenth century.

3. The East and the West Arabs both assigned to the numerals a Hindu origin. "Gubar-numerals" are "dust-numerals," in memory of the Brahmin practice of reckoning on tablets strewn with dust or sand.

4. Not less startling was the fact that both sets of Arabic numerals resembled the apices of Boethius much more closely than the modern Devanagari-numerals. The Gubar-numerals in particular bore a striking resemblance to the apices. But what are the apices? Boethius, a Roman writer of the sixth century, wrote a geometry, in which he speaks of an abacus, *which he attributes to the Pythagoreans.* Instead of following the ancient practice of using pebbles on the abacus, he employed the apices, which were probably small cones. Upon each of these was drawn one of the nine digits, now called "apices." These digits occur again in the body of the text.[1] *Boethius gives no symbol for zero.*

Need we marvel that, in attempting to harmonize these apparently incongruous facts, scholars for a long time failed to agree on an explanation of the strange metamorphoses of the numerals, or the course of their fleeting footsteps, as they migrated from land to land?

The explanation most favourably received is that of Woepcke.[2]

[1] See FRIEDLEIN's edition of BOETHIUS, Leipzig, 1867, p. 397.

[2] See *Journal Asiatique*, 1st half-year, 1863, pp. 69–79 and 514–529. See also CANTOR, Vol. I., p. 711.

1. The Hindus possessed the nine numerals without the zero, as early as the second century after Christ. It is known that about that time a lively commercial intercourse was carried on between India and Rome, by way of Alexandria. There arose an interchange of ideas as well as of merchandise. The Hindus caught glimpses of Greek thought, and the Alexandrians received ideas on philosophy and science from the East.

2. The nine numerals, without the zero, thus found their way to Alexandria, where they may have attracted the attention of the Neo-Pythagoreans. From Alexandria they spread to Rome, thence to Spain and the western part of Africa. [While the geometry of Boethius (unless the passage relating to the apices be considered an interpolation made five or six centuries after Boethius) proves the presence of the digits in Rome in the fifth century, it must be remarked against this part of Woepcke's theory, that he possesses no satisfactory evidence that they were known in Alexandria in the second or third century.]

3. Between the second and the eighth centuries the nine characters in India underwent changes in shape. A prominent Arabic writer, *Albîrûnî* (died 1038), who was in India during many years, remarks that the shape of Hindu numerals and letters differed in different localities and that when (in the eighth century) the Hindu notation was transmitted to the Arabs, the latter selected from the various forms the most suitable. But before the East Arabs thus received the notation, it had been perfected by the invention of the zero and the application of the principle of position.

4. Perceiving the great utility of that Columbus-egg, the zero, the West Arabs borrowed this epoch-making symbol from those in the East, but retained the old forms of the nine numerals, which they had previously received from Rome. The reason for this retention may have been a disinclination to

Sanscrit letters of the II. Century, A.D.										
Apices of Boethius and of the Middle Ages.	1									
Gubar-numerals of the West Arabs.	1									
Numerals of the East Arabs.	1									
Numerals of Maximus Planudes.	1									
Devanagari-numerals.	१									
From the *Mirrour of the World*, printed by Caxton, 1480.	1									
From the Bamberg Arithmetic by Wagner (?), 1483.	I									
From *De Arte Supputandi* by Tonstall, 1522.	I									

[1] The first six lines in this table are copied from the table at the end of CANTOR, Vol. I. The numerals in the Bamberg Arithmetic are taken from FRIEDRICH UNGER, *Die Methodik der Praktischen Arithmetik in Historischer Entwickelung vom Ausgange des Mittelalters bis auf die Gegenwart*, Leipzig, 1888, p. 39. (Hereafter this work will be cited as UNGER.) The double forms for 3, 4, 5, 7, appear in the Bamberg Arithmetic intermixed. Caxton's numerals are taken from W. W. R. BALL'S *History of Mathematics*, 1893, p. 190. For Tonstall's numerals we are indebted to the kindness of Dr. R. Garnett of the British Museum, who copied them from the original.

unnecessary change, coupled, perhaps, with a desire to be contrary to their political enemies in the East.

5. The West Arabs remembered the Hindu origin of the old forms, and called them "Gubar" or "dust" numerals.

6. After the eighth century the numerals in India underwent further changes, and assumed the greatly modified forms of the modern Devanagari-numerals.

The controversy regarding the origin and transmission of our numerals has engaged many minds. Search for information has led to the close consideration of intellectual, commercial, and political conditions among the Hindus, Alexandrian Greeks, Romans, and particularly among the East and West Arabs. We have here an excellent illustration of how mathematico-historical questions may give great stimulus to the study of the history of civilization, and may throw new light upon it.[1]

In the history of the art of counting, the teacher finds emphasized certain pedagogical precepts. We have seen how universal has been the practice of counting on fingers. In place of fingers, groups of other objects were frequently chosen, "as when South Sea Islanders count with cocoanut stalks, putting a little one aside every time they count to 10, and a large one when they come to 100, or when African negroes reckon with pebbles or nuts, and every time they come to 5 put them aside in a little heap."[2] The abstract notion of number is here attained through the agency of *concrete* objects. The arithmetical truths that $2 + 1 = 3$, etc., are here called out by *experience in the manipulation of things.* As we shall see later, the counting by groups of objects in early times led to the invention of the abacus, which is still a valuable school instrument. The earliest arithmetical knowledge of a child

[1] Günther, *op. cit.*, p. 13. [2] Tylor, *op. cit.*, Vol. I., p. 270.

should, therefore, be made to grow out of his experience with different groups of objects; never should he be taught counting by being removed from his toys, and (practically with his eyes closed) made to memorize the abstract statements $1+1=2$, $2+1=3$, etc. Primitive counting in its mode of evolution emphasizes the value of object-teaching.

Recent research[1] seems to indicate that the principle of local value and the use of the zero received systematic and full development in America several centuries earlier than in Asia. The credit of this achievement belongs to the Maya of Central America. About the beginning of the Christian era the Maya possessed a fully developed number-system and chronology. Theirs was not the *decimal*, but the *vigesimal* notation. Their system, as exhibited in their codices, was based on the scale of 20, except in the third position. That is, 20 units of the lowest order (*kins*, or days) make 1 unit of next higher order (*uinals*, or 20 days), 18 uinals make 1 *tun* (360 days, the Maya official year), 20 tuns make 1 *katun* (7200 days), 20 katuns make 1 *cycle*, and 20 cycles make 1 *great cycle*. The numbers 1–19 are expressed by bars and dots. Each bar stands for 5, each dot for 1. Thus, • • is 2, ▬ is 6. The zero is represented by a symbol that looks roughly like a half-closed eye. Accordingly, 37 was expressed by the symbols for 17 (three bars and two dots) in the kin place, and one dot representing 20 in the uinal place, higher up. The numbers were written vertically. To write 360, the Maya drew two zeros, one above the other, with a dot in third place, above the zeros. The largest number found in the codices is 12,489,781.

[1] S. G. Morley, *Introduction to the Study of the Maya Hieroglyphs*, Washington, 1915.

ARITHMETIC AND ALGEBRA

EGYPT

The most ancient mathematical handbook known to our time is a papyrus included in the Rhind collection of the British Museum. This interesting hieratic document, described by Birch in 1868, and translated by Eisenlohr[1] in 1877, was written by an Egyptian, Ahmes by name, in the reign of Ra-ā-us, some time between 1700 and 2000 B.C. It is entitled: "Directions for obtaining the knowledge of all dark things." It claims to be founded on older documents prepared in the time of King [Ra-en-m]āt. Unless specialists are in error regarding the name of this last king, Ra-en-mat, [*i.e.* Amenemhat III.], whose name is not legible in the papyrus, it follows that the original is many centuries older than the copy made by Ahmes. The Ahmes papyrus, therefore, gives us an idea of Egyptian geometry, arithmetic, and algebra as it existed certainly as early as 1700 B.C., and possibly as early as 3000 B.C. While it does not disclose as extensive mathematical knowledge as one might expect to find among the builders of the pyramids, it nevertheless shows in

[1] A. Eisenlohr, *Ein mathematisches Handbuch der alten Aegypter (Papyrus Rhind des British Museum)*, Leipzig, 1877; 2d ed. 1891. See also Cantor, I., pp. 59–84, and James Gow, *A Short History of Greek Mathematics*, Cambridge, 1884, pp. 16–19. The last work will be cited hereafter as Gow.

several particulars a remarkably advanced state of mathematics at the time when Abraham visited Egypt.

From the Ahmes papyrus we infer that the Egyptians knew nothing of theoretical results. It contains no theorems, and hardly any general rules of procedure. In most cases the writer treats in succession several problems of the same kind. From them it would be easy, by induction, to obtain general rules, but this is not done. When we remember that only one hundred years ago it was the practice of many English arithmetical writers to postpone the discussion of fractions to the end of their books, it is surprising to find that this hand-book of 4000 years ago *begins* with exercises in fractions, and pays but little attention to whole numbers. Gow probably is right in his conjecture that Ahmes wrote for the *élite* mathematicians of his day.

While fractions occur in the oldest mathematical records which have been found, the ancients nevertheless attained little proficiency in them. Evidently this subject was one of great difficulty. Simultaneous changes in both numerator and denominator were usually avoided. Fractions are found among the Babylonians. Not only had they *sexagesimal* divisions of weights and measures, but also *sexagesimal fractions*.[1] These fractions had a *constant* denominator (60), and were indicated by writing the numerator a little to the right of the ordinary position for a word or number, the denominator being understood. We shall see that the Romans likewise usually kept the denominators constant, but equal to 12. The Egyptians and Greeks, on the other hand, kept the numerators constant and dealt with variable denominators. Ahmes confines himself to fractions of a special class, namely *unit-fractions*, having unity for their numerators. A fraction was designated by

[1] CANTOR, Vol. I., pp. 23, 31.

writing the denominator and then placing over it either a dot or a symbol, called *ro*. Fractional values which could not be expressed by any one unit-fraction were represented by the sum of two or more of them. Thus he wrote $\frac{1}{6}\ \frac{1}{18}$ in place of $\frac{2}{9}$. It is curious to observe that while Ahmes knows $\frac{2}{3}$ to equal $\frac{1}{2}\ \frac{1}{6}$, he makes an exception in this case, adopts a special symbol for $\frac{2}{3}$, and allows it to appear often among the unit-fractions.[1] A fundamental problem in Ahmes's treatment of fractions was, how to find the unit-fractions, the sum of which represents a given fractional value. This was done by aid of a table, given in the papyrus, in which all fractions of the form $\frac{2}{2n+1}$ (n designating successively all integers up to 49) are reduced to the sum of unit-fractions. Thus, $\frac{2}{35} = \frac{1}{30}\ \frac{1}{42}$, $\frac{2}{43} = \frac{1}{42}\ \frac{1}{86}\ \frac{1}{129}\ \frac{1}{301}$. With aid of this table Ahmes could work out problems like these, "Divide 2 by 3," "Divide 2 by 17," etc. Ahmes nowhere states why he confines himself to 2 as the numerator, nor does he inform us how, when, and by whom the table was constructed. It is plain, however, that by the use of this table any fraction whose denominator is odd and less than 100, can be represented as the sum of unit-fractions. The division of 5 by 21 may have been accomplished as follows: $5 = 1 + 2 + 2$. From the table we get $\frac{2}{21} = \frac{1}{14}\ \frac{1}{42}$. Then $\frac{5}{21} = \frac{1}{21} + (\frac{1}{14}\ \frac{1}{42}) + (\frac{1}{14}\ \frac{1}{42}) = \frac{1}{21} + (\frac{2}{14}\ \frac{2}{42}) = \frac{1}{21}\ \frac{1}{7}\ \frac{1}{21} = \frac{1}{7}\ \frac{2}{21} = \frac{1}{7}\ \frac{1}{14}\ \frac{1}{42}$. It may be remarked that there are many ways of breaking up a fraction into unit-fractions, but Ahmes invariably gives only one. Contrary to his usual practice, he gives a general rule for multiplying a fraction by $\frac{2}{3}$. He says: "If you are asked, what is $\frac{2}{3}$ of $\frac{1}{5}$, take the double and the sixfold; that is $\frac{2}{3}$ of it. One must proceed likewise for any other fraction." As only the denominator was written down, he means

[1] Cantor, Vol. I., p. 61.

by the "double" and "sixfold," double and sixfold the denominator. Since $\frac{2}{3} = \frac{1}{2}\ \frac{1}{6}$, the rule simply means

$$\frac{2}{3} \times \frac{1}{5} = \frac{1}{2 \times 5} + \frac{1}{6 \times 5}.$$

His statement "likewise for any other fraction" appears to mean[1]

$$\frac{2}{3} \times \frac{1}{a} = \frac{1}{2a} + \frac{1}{6a}.$$

The papyrus contains 17 examples which show by what a fraction or a mixed number must be multiplied, or what must be added to it, to obtain a given value. The method consists in the reduction of the given fractions to a common denominator. Strange to say, the latter is not always chosen so as to be exactly divisible by all the given denominators. Ahmes gives the example, increase $\frac{1}{4}\ \frac{1}{8}\ \frac{1}{10}\ \frac{1}{30}\ \frac{1}{45}$ to 1. The common denominator taken appears to be 45, for the numbers are stated as $11\frac{1}{4}$, $5\frac{1}{2}\ \frac{1}{8}$, $4\frac{1}{2}$, $1\frac{1}{2}$, 1. The sum of these is $23\ \frac{1}{2}\ \frac{1}{4}\ \frac{1}{8}$ forty-fifths. Add to this $\frac{1}{9}\ \frac{1}{40}$, and the sum is $\frac{2}{3}$. Add $\frac{1}{3}$, and we have 1. Hence the quantity to be added to the given fraction is $\frac{1}{3}\ \frac{1}{9}\ \frac{1}{40}$.

By what must $\frac{1}{16}\ \frac{1}{112}$ be multiplied to give $\frac{1}{8}$? The common denominator taken is 28, then $\frac{1}{16} = \frac{1\frac{1}{2}\ \frac{1}{4}}{28}$, $\frac{1}{112} = \frac{\frac{1}{4}}{28}$, their sum $= \frac{2}{28}$. Again $\frac{1}{8} = \frac{3\frac{1}{2}}{28}$. Since $2 + 1 + \frac{1}{2} = 3\frac{1}{2}$, take first $\frac{2}{28}$, then its half $\frac{1}{28}$, then half of that $\frac{\frac{1}{2}}{28}$, and we have $\frac{3\frac{1}{2}}{28}$. Hence $\frac{1}{16}\ \frac{1}{112}$ becomes $\frac{1}{8}$ on multiplication by $1\frac{1}{2}\ \frac{1}{4}$.

These examples disclose methods quite foreign to modern mathematics.[2] One process, however, found extensive appli-

[1] Cantor, Vol. I., p. 66.

[2] Cantor's account of fractions in the Ahmes papyrus suggested to J. J. Sylvester (then in Baltimore) material for a paper "On a Point in the Theory of Vulgar Fractions," *Am. Journal of Mathematics*, III., 1880, pp. 32 and 388.

cation in arithmetics of the fifteenth century and later, namely, that of aliquot parts, used largely in Practice. In the second of the above examples aliquot parts are taken of $\frac{2}{28}$. This process is seen again in Ahmes's calculations to verify the identities in the table of unit-fractions.

Ahmes then proceeds to eleven problems leading to simple equations with one unknown quantity. The unknown is called *hau* or 'heap.' Symbols are used to designate addition, subtraction, and equality. We give the following specimen:[1]

Hā′	neb-f	ma-f	ro sefeχ-f	hi-f	χeper-f	em sa sefeχ;
Heap	its $\frac{2}{3}$,	its $\frac{1}{2}$,	its $\frac{1}{7}$, its	whole,	it gives	37
i.e.	$x(\frac{2}{3}$	$+\frac{1}{2}$	$+\frac{1}{7}$	$+1)$	$=$	37

Here stands for $\frac{2}{3}$, for $\frac{1}{2}$. Other unit-fractions are indicated by writing the number and placing , *ro*, above it. A problem resembling the one just given reads: "Heap, its $\frac{2}{3}$, its $\frac{1}{2}$, its $\frac{1}{7}$, its whole, it makes 33;" *i.e.* $\frac{2}{3}x+\frac{x}{2}+\frac{x}{7}+x=33$. The solution proceeds as follows: $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}\ x=33$. Then $1\ \frac{2}{3}\ \frac{1}{2}\ \frac{1}{7}$ is multiplied in the manner sketched above, so as to get 33, thereby obtaining *heap* equal to $14\ \frac{1}{4}\ \frac{1}{97}\ \frac{1}{56}\ \frac{1}{679}\ \frac{1}{776}\ \frac{1}{1164}\ \frac{1}{3492}$. Here, then, we have the solution of an algebraic equation!

In this Egyptian document, as also among early Babylonian records, are found examples of arithmetical and geometrical progressions. Ahmes gives an example: "Divide 100 loaves among 5 persons; $\frac{1}{7}$ of what the first three get is what the last two get. What is the difference?"[2] Ahmes gives the solution: "Make the difference $5\frac{1}{2}$; 23, $17\frac{1}{2}$, 12, $6\frac{1}{2}$, 1. Multiply

[1] Ludwig Matthiessen, *Grundzüge der Antiken und Modernen Algebra der litteralen Gleichungen*, Leipzig, 1878, p. 269. Hereafter we cite this work as Matthiessen.

[2] Cantor, Vol. I., p. 78.

by $1\frac{2}{3}$; $38\frac{1}{3}$, $29\frac{1}{6}$, 20, 10 $\frac{2}{3}$ $\frac{1}{6}$, $1\frac{2}{3}$." How did Ahmes come upon $5\frac{1}{2}$? Perhaps thus:[1] Let a and $-d$ be the first term and the difference in the required arithmetical progression, then $\frac{1}{7}[a+(a-d)+(a-2d)] = (a-3d)+(a-4d)$, whence $d = 5\frac{1}{2}(a-4d)$, *i.e.* the difference d is $5\frac{1}{2}$ times the last term. *Assuming* the last term $= 1$, he gets his first progression. The sum is 60, but should be 100; hence multiply by $1\frac{2}{3}$, for $60 \times 1\frac{2}{3} = 100$. We have here a method of solution which appears again later among the Hindus, Arabs, and modern Europeans — the method of *false position*. It will be explained more fully elsewhere.

Still more curious is the following in Ahmes. He speaks of a *ladder* consisting of the numbers 7, 49, 343, 2401, 16807. Adjacent to these powers of 7 are the words *picture*, *cat*, *mouse*, *barley*, *measure*. Nothing in the papyrus gives a clue to the meaning of this, but Cantor thinks the key to be found in the following problem occurring 3000 years later in the *liber abaci* (1202 A.D.) of Leonardo of Pisa: 7 old women go to Rome; each woman has 7 mules, each mule carries 7 sacks, each sack contains 7 loaves, with each loaf are 7 knives, each knife is put in 7 sheaths. What is the sum total of all named? This has suggested the following wording in Ahmes: 7 persons have each 7 cats; each cat eats 7 mice, each mouse eats 7 ears of barley, from each ear 7 measures of corn may grow. What is the series arising from these data, what the sum of its terms? Ahmes gives the numbers, also their sum, 19607. Problems of this sort may have been proposed for amusement. If the above interpretations are correct, it looks as if "mathematical recreations" were indulged in by scholars forty centuries ago.

In the *hau*-problems, of which we gave one example, we see the beginnings of algebra. So far as documentary evidence

[1] CANTOR, Vol. I., p. 78.

goes, arithmetic and algebra are coeval. That arithmetic is actually the older there can be no doubt. But mark the close relation between them at the very beginning of authentic history. So in mathematical teaching there ought to be an intimate union between the two. In the United States algebra was for a long time set aside, while extraordinary emphasis was laid on arithmetic. The readjustment has come. The "Mathematical Conference of Ten" of 1892 voiced the sentiment of the best educators when it recommended the earlier introduction of certain parts of elementary algebra.

The part of Ahmes's papyrus which has to the greatest degree taxed the ingenuity of specialists is the table of unit-fractions. How was it constructed? Some hold that it was not computed by any one person, nor even in one single epoch, and that the method of construction was not the same for all fractions. On the other hand Loria thinks he has discovered a general mode by which this and similar existing tables may have been calculated.[1]

That the period of Ahmes was a flowering time for Egyptian mathematics appears from the fact that there exist two other papyri of this period, containing quadratic equations! They were found at Kahun, south of the pyramid of Illahun. These documents bear close resemblance to Ahmes's, as does also the Akhmim papyrus,[2] recently discovered at Akhmim, a city on

[1] See the following articles by Gino Loria: "Congetture e ricerche sull' aritmetica degli antichi Egiziani" in *Bibliotheca Mathematica*, 1892, pp. 97–109; "Un nuovo documento relativo alla logistico greco-egiziana," *Ibid.*, 1893, pp. 79–89; "Studi intorno alla logistica greco-egiziana," *Estratto dal Volume XXXII* (1° *della* 2ª *serie*) *del Giornale di Mathematiche di Battaglini*, pp. 7–35.

[2] J. Baillet, "Le papyrus mathématique d'Akhmim," *Mémoires publiés par les membres de la mission archéologique française au Caire, T. IX.*, 1ʳ fascicule, Paris, 1892, pp. 1–88. See also Loria's papers in the preceding note, and Cantor, Vol. I., 1907, pp. 95, 96.

the Nile in Upper Egypt. It is in Greek and is supposed to have been written at some time between 500 and 800 A.D. Like his ancient predecessor Ahmes, its author gives tables of unit-fractions. It marks no progress over the arithmetic of Ahmes. For more than two thousand years Egyptian mathematics was *stationary !*

GREECE

In passing to Greek arithmetic and algebra, we first observe that the early Greeks were not automaths; they acknowledged the Egyptian priests to have been their teachers. While in geometry the Greeks soon reached a height undreamed of by the Egyptian mind, they contributed hardly anything to the art of calculation. Not until the golden period of geometric discovery had passed away, do we find in Nicomachus and Diophantus substantial contributors to algebra.

Greek mathematicians were in the habit of discriminating between the science of numbers and the art of computation. The former they called *arithmetica*, the latter *logistica*.

Greek writers seldom refer to calculation with alphabetic numerals. Addition, subtraction, and even multiplication were probably performed on the abacus. Eutocius, a commentator of the sixth century A.D., exhibits a great many multiplications, such as expert Greek mathematicians of classical time may have used.[1] While among the Sophists computation received some attention, it was pronounced a vulgar and childish art by Plato, who cared only for the philosophy of arithmetic.

Greek writers did not confine themselves to unit-fractions as closely as did the Egyptians. Unit-fractions were designated by simply writing the denominator with a double accent.

[1] For specimens of such multiplications see CANTOR, *Beiträge z. Kulturl. d. Völker*, p. 393; HANKEL, p. 56; GOW, p. 50; FRIEDLEIN, p. 76; my *History of Mathematics*, 1895, p. 65.

Thus, $\rho\iota\beta'' = \frac{1}{112}$. Other fractions were usually indicated by writing the numerator once with an accent and the denominator twice with a double accent. Thus, $\iota\zeta'\kappa\alpha''\kappa\alpha'' = \frac{17}{21}$. As with the Egyptians, unit-fractions in juxtaposition are to be added.

Like the Eastern nations, the Egyptians and Greeks employ two aids to computation, the abacus and finger symbolism. It is not known what the signs used in the latter were, but by the study of ancient statuary, bas-reliefs, and paintings this secret may yet be unravelled. Of the abacus there existed many forms at different times and among the various nations. In all cases a plane was divided into regions and a pebble or other object represented a different value in different regions. We possess no detailed information regarding the Egyptian or the Greek abacus. Herodotus (II., 36) says that the Egyptians "*calculate with pebbles* by moving the hand from right to left, while the Hellenes move it from left to right." This indicates a primitive and instrumental mode of counting with aid of pebbles. The fact that the hand was moved towards the right, or towards the left, indicates that the plane or board was divided by lines which were *vertical*, *i.e.* up and down, with respect to the computor. Iamblichus informs us that the abacus of the Pythagoreans was a board strewn with dust or sand. In that case, any writing could be easily erased by sprinkling the board anew. A pebble placed in the right hand space or column designated 1, if placed in the second column from the right 10, if in the third column 100, etc. Probably, never more than nine pebbles were placed in one column, for ten of them would equal one unit of the next higher order. The Egyptians on the other hand chose the column on the extreme left as the place for units, the second column from the left designating tens, the third hundreds, etc. In further support of this description of the abacus, a comparison attrib-

uted by Diogenes Laertius (I., 59) to Solon is interesting:[1] "A person friendly with tyrants is like the stone in computation, which signifies now much, now little."

The abacus appears to have been used in Egypt and Greece to carry out the simpler calculation with integers. The handbook of Ahmes with its treatment of fractions was presumably written for those already familiar with abacal or digital reckoning. Greek mathematical works usually give the numerical results without exhibiting the computation itself. Thus advanced mathematicians frequently had occasion to extract the square root. In his *Mensuration of the Circle*, Archimedes states, for instance, that $\sqrt{3} < \frac{1351}{780}$ and $\sqrt{3} > \frac{265}{153}$, but he gives no clue to his method of approximation.[2]

When sexagesimal numbers (introduced from Babylonia into Greece about the time of the Greek geometer Hypsicles and the Alexandrian astronomer Ptolemæus) were used, then the mode of root-extraction resembled that of the present time. A specimen of the process, as given by Theon, the father of Hypatia, has been preserved. He finds $\sqrt{4500^\circ} = 67^\circ\ 4'\ 55''$.

Archimedes showed how the Greek system of numeration might be extended so as to embrace numbers as large as you please. By the ordinary nomenclature of his day, numbers could be expressed up to 10^8. Taking this 10^8 as a unit of second order, 10^{16} as one of the third order, etc., the system may be sufficiently extended to enable one to count the very sands. Assuming 10,000 grains of sand to fill the space of a poppy-seed, he finds a number which would exceed the number

[1] Cantor, Vol. I., p. 132.

[2] What the Archimedean and, in general, the Greek method of root-extraction really was, has been a favourite subject of conjecture. See for example, H. Weissenborn's *Berechnung des Kreisumfanges bei Archimedes und Leonardo Pisano*, Berlin, 1894. For bibliography of this subject, see S. Günther, *Gesch. d. antiken Naturwissenschaft u. Philosophie*, p. 16.

of grains in a sphere whose radius extends from the earth to the fixed stars. A counterpart of this interesting speculation, called the "sand-counter" (arenarius), is found in a calculation attributed to Buddha, the Hindu reformer, of the number of primary atoms in a line one mile in length, when the atoms are placed one against another.

The science of numbers, as distinguished from the art of calculation, commanded the lively attention of the Pythagoreans. Pythagoras himself had imbibed Egyptian mathematics and mysticism. Aside from the capital discovery of irrational quantities (spoken of elsewhere) no very substantial contribution was made by the Pythagoreans to the science of numbers. We may add that by the Greeks irrationals were not classified as numbers. The Pythagoreans sought the origin of all things in numbers; harmony depended on musical proportion; the order and beauty of the universe have their origin in numbers; in the planetary motions they discerned a wonderful "harmony of the spheres." Moreover, some numbers had extraordinary attributes. Thus, *one* is the essence of things; *four* is the most perfect number, corresponding to the human soul. According to Philolaus, 5 is the cause of colour, 6 of cold, 7 of mind, health, and light, 8 of love and friendship.[1] Even Plato and Aristotle refer the virtues to numbers. While these speculations in themselves were fantastic and barren, lines of fruitful mathematical inquiry were suggested by them.

The Pythagoreans classified numbers into odd and even, and observed that the sum of the odd numbers from 1 to $2n+1$ was always a perfect square. Of no particular value were their classifications of numbers into heteromecic, triangular, perfect, excessive, defective, amicable.[2] The Pythag-

[1] Gow, p. 69.

[2] For their definitions see Gow, p. 70, or Cajori, *op. cit.*, p. 68.

oreans paid much attention to the subject of proportion. The quantities $a, b, c, d,$ were said to be in *arithmetical* proportion, when $a-b=c-d$; in *geometrical* proportion, when $a : b = c : d$; in *harmonic* proportion, when $a - b : b - c = a : c$; in *musical* proportion, when $a : \frac{1}{2}(a + b) = 2ab/(a + b) : b$. Iamblichus says that the last was introduced from Babylon.

The 7th, 8th, and 9th books of Euclid's *Elements* are on the science of number, but the 2d and 10th, though professedly geometrical and treating of magnitudes, are applicable to numbers. Euclid was a geometer through and through, and even his arithmetical books smack of geometry. Consider, for instance, definition 21, book VII.[1] "Plane and solid numbers are similar when their sides are proportional." Again, numbers are not written in numerals, nor are they designated by anything like our modern algebraic notation; they are represented by *lines*. This symbolism is very unsuggestive. Frequently properties which our notation unmasks at once could be extracted from these lines only through a severe process of reasoning.[2]

In the 7th book we encounter for the first time a definition of prime numbers. Euclid finds the G. C. D. of two numbers by a procedure identical with our method by division. He applies to numbers the theory of proportion which in the 5th book is developed for magnitudes in general. The 8th book deals with numbers in continued proportion. The 9th book finishes that subject, deals with primes, and contains the proof for the remarkable theorem 20, that the number of primes is infinite.

During the four centuries after Euclid, geometry monopolized the attention of the Greeks and the theory of numbers

[1] HEIBERG'S edition, Vol. II., p. 189.

[2] G. H. F. NESSELMANN, *Die Algebra der Griechen*, Berlin, 1842, p. 184. To be cited hereafter as NESSELMANN.

was neglected. Of this period only two names deserve mention, *Eratosthenes* (about 275–194 B.C.) and *Hypsicles* (between 200 and 100 B.C.). To the latter we owe researches on polygonal numbers and arithmetical progressions. Eratosthenes invented the celebrated "sieve" for finding prime numbers. Write down in succession all odd numbers from 3 up. By erasing every third number after 3, sift out all multiples of 3; by erasing every fifth number after 5, sift out all multiples of 5, and so on. The numbers left after this sifting are all prime. While the invention of the "sieve" called for no great mental powers, it is remarkable that after Eratosthenes no advance was made in the mode of finding the primes, nor in the determination of the number of primes which exist in the numerical series 1, 2, 3, ... n, until the nineteenth century, when Gauss, Legendre, Dirichlet, Riemann, and Chebîchev enriched the subject with investigations mostly of great difficulty and complexity.

The study of arithmetic was revived about 100 A.D. by *Nicomachus*,[1] a native of Gerasa (perhaps a town in Arabia) and known as a Pythagorean. He wrote in Greek a work entitled *Introductio Arithmetica*. The historical importance of this work is great, not so much on account of original matter therein contained, but because it is (so far as we know) the earliest systematic text-book on arithmetic, and because for over 1000 years it set the fashion for the treatment of this subject in Europe. In a small measure, Nicomachus did for arithmetic what Euclid did for geometry. His arithmetic was as famous in his day as was, later, Adam Riese's in Germany, and Cocker's in England. Wishing to compliment a computor, Lucian says, "You reckon like Nicomachus of Gerasa."[2] The

[1] See Nesselmann, pp. 191–216; Gow, pp. 88–95; Cantor, Vol. I., pp. 400–404.

[2] Quoted by Gow (p. 89) from *Philopatris*, 12.

work was brought out in a Latin translation by Appuleius (now lost) and then by Boethius. In Boethius's translation the elementary parts of the work were in high authority in Western Europe until the country was invaded by Hindu arithmetic. Thereupon for several centuries Greek arithmetic bravely but vainly struggled for existence against its immeasurably superior Indian rival.

The style of Nicomachus differs essentially from that of his predecessors. It is not deductive, but inductive. The geometrical style is abandoned; the different classes of numbers are exhibited in actual numerals. The author's main object is classification. Being under the influence of philosophy and theology, he sometimes strains a point to secure a division into groups of three. Thus, *odd* numbers are either "prime and uncompounded," "compounded," or "compounded but prime to one another." His nomenclature resulting from this classification is exceedingly burdensome. The Latin equivalents for his Greek terms are found in the *printed* arithmetics of his disciples, 1500 years later. Thus the ratio $\frac{m+1}{m}$ is *superparticularis*, $\frac{m}{m+1}$ is *subsuperparticularis*, $3\frac{1}{4} = \frac{3 \times 4 + 1}{4}$ is *triplex sesquiquartus*.[1] Nicomachus gives tables of numbers in form of a chess-board of 100 squares. It might have answered as a multiplication-table, but it appears to have been used in the study of ratios.[2] He describes polygonal numbers, the different kinds of proportions (11 in all), and treats of the summation of numerical series. To be noted is the absence of rules of computation, of problem-working, and of practical arithmetic. He gives the following important proposition. All cubical numbers are equal to the sum of successive odd

[1] Gow, pp. 90, 91.

[2] Friedlein, p. 78; Cantor, Vol. I., p. 431.

numbers. Thus, $8 = 2^3 = 3 + 5$; $27 = 3^3 = 7 + 9 + 11$; $64 = 4^3 = 13 + 15 + 17 + 19$.

In the writings of Nicomachus, Iamblichus, Theon of Smyrna, Thymaridas, and others, are found investigations algebraic in their nature. Thymaridas in one place uses a Greek word meaning "unknown quantity" in a manner suggesting the near approach of algebra. Of interest in tracing the evolution of algebra are the arithmetical epigrams in the *Palatine Anthology*, which contained about 50 problems leading to linear equations.[1] Before the introduction of algebra, these problems were propounded as puzzles. No. 23 gives the times in which four fountains can fill a reservoir separately and requires the time they can fill it conjointly.[2] No. 9. What part of the day has disappeared, if the time left is twice two-thirds of the time passed away? Sometimes included among these epigrams is the famous "cattle-problem," which Archimedes is said to have propounded to the Alexandrian mathematicians.[3] This difficult problem is indeterminate. In the first part of it, from only seven equations, eight unknown quantities in integral numbers are to be found. Gow states it thus: The sun had a herd of bulls and cows, of different colours. (1) Of Bulls, the white (W) were in number $(\frac{1}{2} + \frac{1}{3})$ of the blue (B) and yellow (Y); the B were $(\frac{1}{4} + \frac{1}{5})$ of the Y and piebald (P); the P were $(\frac{1}{6} + \frac{1}{7})$ of the W and Y. (2) Of Cows, which had the same colours (w, b, y, p), $w = (\frac{1}{3} + \frac{1}{4})(B + b)$; $b = (\frac{1}{4} + \frac{1}{5})(P + p)$; $p = (\frac{1}{5} + \frac{1}{6})(Y + y)$; $y = (\frac{1}{6} + \frac{1}{7})(W + w)$.

[1] These epigrams were written in Greek, perhaps about the time of Constantine the Great. For a German translation, see G. Wertheim, *Die Arithmetik und die Schrift über Polygonalzahlen des Diophantus von Alexandria*, Leipzig, 1890, pp. 330–344.

[2] Wertheim, *op. cit.*, p. 337.

[3] Whether it originated at the time of Archimedes or later is discussed by T. L. Heath, *Diophantos of Alexandria*, Cambridge, 1885, pp. 142–147.

Find the number of bulls and cows. This leads to excessively high numbers, but to add to its complexity, a second series of conditions is superadded, leading to an indeterminate equation of the second degree.

Most of the problems in the *Palatine Anthology*, though puzzling to an arithmetician, are easy to an algebraist. Such problems became popular about the time of Diophantus and doubtless acted as a powerful mental stimulus.

Diophantus, one of the last Alexandrian mathematicians, is generally regarded as an algebraist of great fertility.[1] He died about 330 A.D. His age was 84, as is known from an epitaph to the following effect: Diophantus passed $\frac{1}{6}$ of his life in childhood, $\frac{1}{12}$ in youth, and $\frac{1}{7}$ more as a bachelor; five years after his marriage, was born a son who died four years before his father, at half his father's age. This epitaph states about all we know of Diophantus. We are uncertain as to the time of his death and ignorant of his parentage and place of nativity. Were his works not written in Greek, no one would suspect them of being the product of Greek mind. The spirit pervading his masterpiece, the *Arithmetica* [said to have been written in thirteen books, of which only six (seven ?)[2] are extant] is as different from that of the great classical works of the time of Euclid as pure geometry is from pure analysis. Among the Greeks, Diophantus had no prominent forerunner, no prominent disciple. Except for his works, we should be obliged to say that the Greek mind accomplished nothing notable in algebra. Before the discovery of the Ahmes papyrus, the *Arithmetica* of Diophantus was the oldest known work on algebra. Diophantus introduces the notion of an algebraic equation expressed in symbols. Being completely

[1] "How far was Diophantos original?" see HEATH, *op. cit.*, pp. 133–159.

[2] CANTOR, Vol. I., pp. 466, 467.

divorced from geometry, his treatment is purely analytical. He is the first to say that "a number to be subtracted multiplied by a number to be subtracted gives a number to be added." This is applied to differences, like $(2x-3)(2x-3)$, the product of which he finds without resorting to geometry. Identities like $(a+b)^2=a^2+2ab+b^2$, which are elevated by Euclid to the exalted rank of geometric theorems, with Diophantus are the simplest consequences of algebraic laws of operation. Diophantus represents the unknown quantity x by ς', the square of the unknown x^2 by $\delta^{\hat{\upsilon}}$, x^3 by $\kappa^{\hat{\upsilon}}$, x^4 by $\delta\delta^{\hat{\upsilon}}$. His sign for subtraction is $\pitchfork$, his symbol for equality, ι. Addition is indicated by juxtaposition. Sometimes he ignores these symbols and describes operations in words, when the symbols would have answered better. In a polynomial all the positive terms are written before any of the negative ones. Thus, x^3-5x^2+8x-1 would be in his notation,[1] $\kappa^{\hat{\upsilon}}\bar{\alpha}\varsigma^{\hat{o}\iota}\bar{\eta}\ \pitchfork\ \delta^{\hat{\upsilon}}\bar{\epsilon}\mu^{\hat{o}}\bar{\alpha}$. Here the numerical coefficient follows the x.

To be emphasized is the fact that in Diophantus the fundamental algebraic conception of *negative* numbers is wanting. In $2x-10$ he avoids as absurd all cases where $2x<10$. Take *Probl.* 16 *Bk.* I. in his *Arithmetica:* "To find three numbers such that the sums of each pair are given numbers." If a, b, c are the given numbers, then one of the required numbers is $\frac{1}{2}(a+b+c)-c$. If $c>\frac{1}{2}(a+b+c)$, then this result is unintelligible to Diophantus. Hence he imposes upon the problem the limitation, "But half the sum of the three given numbers must be greater than any one singly." Diophantus does not give solutions general in form. In the present instance the special values 20, 30, 40 are assumed as the given numbers.

In problems leading to simultaneous equations Diophantus adroitly uses only one symbol for the unknown quantities.

[1] Heath, *op. cit.*, p. 72.

This poverty in notation is offset in many cases merely by skill in the selection of the unknown. Frequently he follows a method resembling somewhat the Hindu "false position": a preliminary value is assigned to the unknown which satisfies only one or two of the necessary conditions. This leads to expressions palpably wrong, but nevertheless suggesting some stratagem by which one of the correct values can be obtained.[1]

Diophantus knows how to solve quadratic equations, but in the extant books of his *Arithmetica* he nowhere explains the mode of solution. Noteworthy is the fact that he always gives but one of the two roots, even when both roots are positive. Nor does he ever accept as an answer to a problem a quantity which is negative or irrational.

Only the first book in the *Arithmetica* is devoted to determinate equations. It is in the solution of indeterminate equations (of the second degree) that he exhibits his wonderful inventive faculties. However, his extraordinary ability lies less in discovering general methods than in reducing all sorts of equations to particular forms which he knows how to solve. Each of his numerous and various problems has its own distinct method of solution, which is often useless in the most closely related problem. "It is, therefore, difficult for a modern mathematician, after studying 100 Diophantine solutions, to solve the 101st. . . . Diophantus dazzles more than he delights."[2]

The absence in Diophantus of general methods for dealing with indeterminate problems compelled modern workers on this subject, such as Euler, Lagrange, Gauss, to begin anew.

[1] Gow, pp. 110, 116, 117.

[2] HANKEL, p. 165. It should be remarked that HEATH, *op. cit.*, pp. 83–120, takes exception to Hankel's verdict and endeavours to give a general account of Diophantine methods.

Diophantus could teach them nothing in the way of general methods. The result is that the modern theory of numbers is quite distinct and a decidedly higher and nobler science than Diophantine Analysis. Modern disciples of Diophantus usually display the weaknesses of their master and for that reason have failed to make substantial contributions to the subject.

Of special interest to us is the method followed by Diophantus in solving a linear determinate equation. His directions are: "If now in any problem the same powers of the unknown occur on both sides of the equation, but with different coefficients, we must subtract equals from equals until we have one term equal to one term. If there are on one side, or on both sides, terms with negative coefficients, these terms must be added on both sides, so that on both sides there are only positive terms. Then we must again subtract equals from equals until there remains only one term in each number."[1] Thus what is nowadays achieved by transposing, simplifying, and dividing by the coefficient of x, was accomplished by Diophantus by addition and subtraction. It is to be observed that in Diophantus, and in fact in all writings of antiquity, the conception of a *quotient* is wanting. An operation of *division* is nowhere exhibited. When one number had to be divided by another, the answer was reached by repeated subtractions.[2]

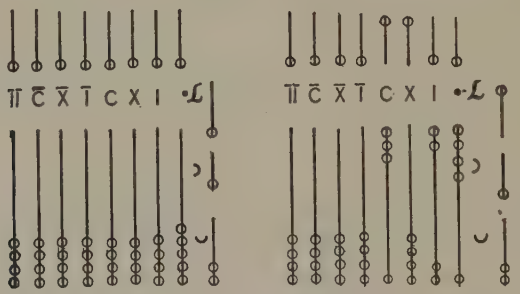
ROME

Of Roman methods of computation more is known than of Greek or Egyptian. Abacal reckoning was taught in schools. Writers refer to pebbles and a dust-covered abacus, ruled into columns. An Etruscan (?) relic, now preserved in Paris, shows a computor holding in his left hand an abacus with

[1] Wertheim's *Diophantus*, p. 7.
[2] Nesselmann, p. 112; Friedlein, p. 79.

numerals set in columns, while with the right hand he lays pebbles upon the table.[1]

The Romans used also another kind of abacus, consisting of a metallic plate having grooves with movable buttons. By its use all integers between 1 and 9,999,999, as well as some fractions, could be represented. In the two adjoining figures (taken from Fig. 21 in *Friedlein*) the lines represent grooves



and the circles buttons. The Roman numerals indicate the value of each button in the corresponding groove below, the button in the shorter groove above having a fivefold value. Thus $\overline{\text{II}} = 1{,}000{,}000$; hence each button in the long left-hand groove, when in use, stands for 1,000,000, and the button in the short upper groove stands for 5,000,000. The same holds for the other grooves labelled by Roman numerals. The eighth long groove from the left (having 5 buttons) represents duodecimal fractions, each button indicating $\frac{1}{12}$, while the button above the dot means $\frac{6}{12}$. In the ninth column the upper button represents $\frac{1}{24}$, the middle $\frac{1}{48}$, and two lower each $\frac{1}{72}$. Our first figure represents the positions of the buttons before the operation begins; our second figure stands for the number $852\,\frac{1}{8}\,\frac{1}{24}$. The eye has here to distinguish the buttons in use and those left idle. Those counted are one button above

[1] Cantor, Vol. I., p. 529.

c (= 500), and three buttons below c (= 300); one button above x (= 50); two buttons below I (= 2); four buttons indicating duodecimals (= $\frac{1}{3}$); and the button for $\frac{1}{24}$.

Suppose now that 10,318 $\frac{1}{4}$ $\frac{1}{3}$ $\frac{1}{48}$ is to be added to 852 $\frac{1}{3}$ $\frac{1}{24}$. The operator could begin with the highest units, or the lowest units, as he pleased. Naturally the hardest part is the addition of the fractions. In this case the button for $\frac{1}{48}$, the button above the dot and three buttons below the dot were used to indicate the sum $\frac{3}{4}$ $\frac{1}{48}$. The addition of 8 would bring all the buttons above and below 1 into play, making 10 units. Hence, move them all back and move up one button in the groove below x. Add 10 by moving up another of the buttons below x; add 300 to 800 by moving back all buttons above and below c, except one button below, and moving up one button below $\bar{\text{I}}$; add 10,000 by moving up one button below $\bar{\text{x}}$. In subtraction the operation was similar.

Multiplication could be carried out in several ways. In case of 38 $\frac{1}{2}$ $\frac{1}{24}$ times 25 $\frac{1}{3}$, the abacus may have shown successively the following values: 600 (= 30 · 20), 760 (= 600 + 20 · 8), 770 (= 760 + $\frac{1}{2}$ · 20), 770 $\frac{10}{12}$ (= 770 + $\frac{1}{24}$ · 20), 920 $\frac{10}{12}$ (= 770 $\frac{10}{12}$ + 30 · 5), 960 $\frac{10}{12}$ (= 920 $\frac{10}{12}$ + 8 · 5), 963 $\frac{1}{3}$ (= 960 $\frac{10}{12}$ + $\frac{1}{2}$ · 5), 963 $\frac{1}{2}$ $\frac{1}{24}$ (= 963 $\frac{1}{3}$ + $\frac{1}{24}$ · 5), 973 $\frac{1}{2}$ $\frac{1}{24}$ (= 963 $\frac{1}{2}$ $\frac{1}{24}$ + $\frac{1}{3}$ · 30), 976 $\frac{2}{12}$ $\frac{1}{24}$ (= 973 $\frac{1}{2}$ $\frac{1}{24}$ + 8 · $\frac{1}{3}$), 976 $\frac{1}{3}$ $\frac{1}{24}$ (= 976 $\frac{2}{12}$ $\frac{1}{24}$ + $\frac{1}{2}$ · $\frac{1}{3}$), 976 $\frac{1}{3}$ $\frac{1}{24}$ $\frac{1}{72}$ (= 976 $\frac{1}{3}$ $\frac{1}{24}$ + $\frac{1}{3}$ · $\frac{1}{24}$).[1]

In division the abacus was used to represent the remainder resulting from the subtraction from the dividend of the divisor or of a convenient multiple of the divisor. The process was complicated and difficult. These methods of abacal computation show clearly how multiplication or division can be carried out by a series of successive additions or subtractions. In this connection we suspect that recourse was had to mental

[1] Friedlein, p. 80.

operations and to the multiplication table. Finger-reckoning may also have been used. In any case the multiplication and division with large numbers must have been beyond the power of the ordinary computor. This difficulty was sometimes obviated by the use of arithmetical tables from which the required sum, difference, or product of two numbers could be copied. Tables of this sort were prepared by *Victorius* of Aquitania, a writer who is well known for his *canon paschalis*, a rule for finding the correct date for Easter, which he published in 457 A.D. The tables of Victorius contain a peculiar notation for fractions, which continued in use throughout the middle ages.[1] Fractions occur among the Romans most frequently in money computations.

The Roman partiality to duodecimal fractions is to be observed. Why duodecimals and not decimals? Doubtless because the decimal division of weights and measures seemed unnatural. In everyday affairs the division of units into 2, 3, 4, 6 equal parts is the commonest, and duodecimal fractions give easier expressions for these parts. In duodecimals the above parts are $\frac{6}{12}$, $\frac{4}{12}$, $\frac{3}{12}$, $\frac{2}{12}$ of the whole; in decimals these parts are $\frac{5}{10}$, $\frac{3\frac{1}{3}}{10}$, $\frac{2\frac{1}{2}}{10}$, $\frac{1\frac{2}{3}}{10}$. Unlike the Greeks, the Romans dealt with concrete fractions. The Roman *as*, originally a copper coin weighing one pound, was divided into 12 *unciæ*. The abstract fraction $\frac{1}{12}$ was expressed concretely by *deunx* (= *de uncia*, *i.e.*, *as* [1] less *uncia* [$\frac{1}{12}$]); $\frac{5}{12}$ was called *quincunx* (= *quinque* [five] *unciæ*); thus each Roman fraction had a special name. Addition and subtraction of such fractions were easy. Fractional computations were the chief part of arithmetical instruction in Roman schools.[2] Horace, in remembrance, perhaps, of his own school-days, gives the following

[1] Consult FRIEDLEIN, pp. 93–98.

[2] HANKEL, pp. 58, 59.

dialogue between teacher and pupil (*Ars poetica*, V. 326–330): "Let the son of Albinus tell me, if from five ounces [*i.e.* $\frac{5}{12}$] be subtracted one ounce [*i.e.* $\frac{1}{12}$], what is the remainder? Come, you can tell. 'One-third.' Good; you will be able to take care of your property. If one ounce [*i.e.* $\frac{1}{12}$] be added, what does it make? 'One-half.'"

Doubtless the Romans unconsciously hit upon a fine pedagogical idea in their concrete treatment of fractions. Roman boys learned fractions in connection with money, weights, and measures. We conjecture that to them fractions meant more than what was conveyed by the definition, "broken number," given in old English arithmetics.

One of the last Roman writers was *Boethius* (died 524), who is to be mentioned in this connection as the author of a work, *De Institutione Arithmetica*, essentially a translation of the arithmetic of Nicomachus, although some of the most beautiful arithmetical results in the original are omitted by Boethius. The historical importance of this translation lies in the extended use made of it later in Western Europe.

The Roman laws of inheritance gave rise to numerous arithmetical examples. The following is of interest in itself, and also because its occurrence elsewhere at a later period assists us in tracing the source of arithmetical knowledge in Western Europe: A dying man wills, that if his wife, being with child, gives birth to a son, the son shall receive $\frac{2}{3}$, and she $\frac{1}{3}$ of his estate; but if a daughter is born, the daughter shall receive $\frac{1}{3}$, and the wife $\frac{2}{3}$. It happens that twins are born, a boy and a girl. How shall the estate be divided so as to satisfy the will? The celebrated Roman jurist, Salvianus Julianus, decided that it should be divided into seven equal parts, of which four should go to the son, two to the wife and one to the daughter.

Aside from the (probable) improvement of the abacus and

the development of duodecimal fractions, the Romans made no contributions to arithmetic. The algebra of Diophantus was unknown to them. With them as with all nations of antiquity numerical calculations were long and tedious, for the reason that they never possessed the boon of a perfect notation of numbers with its zero and principle of local value.

GEOMETRY AND TRIGONOMETRY

EGYPT AND BABYLONIA

The crude beginnings of empirical geometry, like the art of counting, must be of very ancient origin. We suspect that our earliest records, reaching back to about 2500 B.C., represent comparatively modern thought. (The Ahmes papyrus and the Egyptian pyramids are probably the oldest evidences of geometrical study.) We find it more convenient, however, to begin with Babylonia. Ancient science is closely knitted with superstition. We have proofs that in Babylonia geometrical figures were used in augury.[1] Among these figures are a pair of parallel lines, a square, a figure with a re-entrant angle, and an incomplete figure, believed to represent three concentric triangles with their sides respectively parallel. The accompanying text contains the Sumerian word *tim*, meaning "line," originally "rope"; hence the conjecture that the Babylonians, like the Egyptians, used ropes in measuring distances, and in determining certain angles. The Babylonian sign ✳ is believed to be associated with the division of the circle into six equal parts, and (as the Babylonians divided the circle into 360 degrees) with the origin of the sexagesimal system. That this division into six parts (probably by the sixfold application of the radius) was known in Babylonia, follows from the inspection of the six spokes in the wheel of a royal carriage

[1] Cantor, Vol. I., pp. 45–48.

represented in a drawing found in the remains of Nineveh. Like the Hebrews (1 Kings vii. 23), the Babylonians took the ratio of the circumference to the diameter equal to 3, a decidedly inaccurate value. Of geometrical demonstrations there is no trace. "As a rule, in the Oriental mind the intuitive powers eclipse the severely rational and logical."

We begin our account of Egyptian geometry with the geometrical problems of the Ahmes papyrus, which are found in the middle of the arithmetical matter. Calculations of the solid contents of barns precede the determination of areas.[1] Not knowing the shape of the barns, we cannot verify the correctness of the computations, but in plane geometry Ahmes's figures usually help us. He considers the area of land in the forms of square, oblong, isosceles triangle, isosceles trapezoid, and circle. Example No. 44 gives 100 as the area of a square whose sides are 10. In No. 51 he draws an isosceles triangle whose sides are 10 ruths and whose base is 4 ruths, and finds the area to be 20. The correct value is 19.6^{-}. Ahmes's approximation is obtained by taking the product of one leg and half the base. The same error occurs in the area of the isosceles trapezoid. Half the sum of the two bases is multiplied by one leg. His treatment of the circle is an actual *quadrature*, for it teaches how to find a square equivalent to the circular area. He takes as the side, the diameter diminished by $\frac{1}{9}$ of itself. This is a fair approximation, for, if the radius is taken as unity, then the side of the square is $1\frac{6}{9}$, and its area $(1\frac{6}{9})^2 = 3.1604\ldots$ Besides these problems there are others relating to pyramids and disclosing some knowledge of similar figures, of proportion, and, perhaps, of rudimentary trigonometry.[2]

Besides the Ahmes papyrus, proofs of the existence of

[1] Gow, pp. 126–130.

[2] Consult CANTOR, Vol. I., pp. 99–102; Gow, p. 128.

ancient Egyptian geometry are found in figures on the walls of old structures. The wall was ruled with squares, or other rectilinear figures, within which coloured pictures were drawn.[1]

The Greek philosopher Democritus (about 460–370 B.C.) is quoted as saying that "in the construction of plane figures . . . no one has yet surpassed me, not even the so-called *Harpedonaptæ* of Egypt." Cantor has pointed out the meaning of the word "harpedonaptæ" to be "rope-stretchers."[2] This, together with other clues, led him to the conclusion that in laying out temples the Egyptians determined by accurate astronomical observation a north and south line; then they constructed a line at right angles to this by means of a rope stretched around three pegs in such a way that the three sides of the triangle formed are to each other as 3 : 4 : 5, and that one of the legs of this right triangle coincided with the N. and S. line. Then the other leg gave the E. and W. line for the exact orientation of the temple. According to a leathern document in the Berlin Museum, "rope-stretching" occurred at the very early age of Amenemhat I. If Cantor's explanation is correct, it follows, therefore, that the Egyptians were familiar with the well-known property of the right-triangle, in case of sides in the ratio 3 : 4 : 5, as early as 2000 B.C.

From what has been said it follows that Egyptian geometry flourished at a very early age. What about its progress in subsequent centuries? In 257 B.C. was laid out the temple of Horus, at Edfu, in Upper Egypt. About 100 B.C. the number of pieces of land owned by the priesthood, and their areas, were inscribed upon the walls in hieroglyphics. The incorrect formula of Ahmes for the isosceles trapezoid is here applied for any trapezium, however irregular. Thus the formulæ of more than 2000 B.C. yield closer approximations than those

[1] Consult the drawings reproduced in CANTOR, Vol. I., p. 108.
[2] Ibidem, p. 104.

written two centuries after Euclid! The conclusion irresistibly follows that the Egyptians resembled the Chinese in the *stationary character*, not only of their government, but also of their science. An explanation of this has been sought in the fact that their discoveries in mathematics, as also in medicine, were entered at an early time upon their sacred books, and that, in after ages, it was considered heretical to modify or augment anything therein. Thus the books themselves closed the gates to progress.

Egyptian geometry is mainly, though not entirely, a geometry of areas, for the measurement of figures and of solids constitutes the main part of it. This practical geometry can hardly be called a *science*. In vain we look for theorems and proofs, or for a logical system based on axioms and postulates. It is evident that some of the rules were purely empirical.

If we may trust the testimony of the Greeks, Egyptian geometry had its origin in the surveying of land necessitated by the frequent overflow of the Nile.

GREECE

About the seventh century B.C. there arose between Greece and Egypt a lively intellectual as well as commercial intercourse. Just as Americans in our time go to Germany to study, so early Greek scholars visited the land of the pyramids. Thales, Œnopides, Pythagoras, Plato, Democritus, Eudoxus, all sat at the feet of the Egyptian priests for instruction. While Greek culture is, therefore, not primitive, it commands our enthusiastic admiration. The speculative mind of the Greek at once transcended questions pertaining merely to the practical wants of everyday life; it pierced into the ideal relations of things, and revelled in the study

of science *as* science. For this reason Greek geometry will always be admired, notwithstanding its limitations and defects.

Eudemus, a pupil of Aristotle, wrote a history of geometry. This history has been lost; but an abstract of it, made by Proclus in his commentaries on Euclid, is extant, and is the most trustworthy information we have regarding early Greek geometry. We shall quote the account under the name of *Eudemian Summary*.

I. *The Ionic School.* — The study of geometry was introduced into Greece by *Thales* of Miletus (640–546 B.C.), one of the "seven wise men." Commercial pursuits brought him to Egypt; intellectual pursuits, for a time, detained him there. Plutarch declares that Thales soon excelled the priests, and amazed King Amasis by measuring the heights of the pyramids from their shadows. According to Plutarch this was done thus: The length of the shadow of the pyramid is to the shadow of a vertical staff, as the unknown height of the pyramid is to the known length of the staff. But according to Diogenes Laertius the measurement was different: The height of the pyramid was taken equal to the length of its shadow at the moment when the shadow of a vertical staff was equal to its own length.[1]

[1] The first method implies a knowledge of the proportionality of the sides of equiangular triangles, which some critics are unwilling to grant Thales. The rudiments of proportion were certainly known to Ahmes and the builders of the pyramids. Allman grants Thales this knowledge, and, in general, assigns to him and his school a high rank. See *Greek Geometry from Thales to Euclid*, by George Johnston Allman, Dublin, 1889, p. 14. Gow (p. 142) also believes in Plutarch's narrative, but Cantor (I., p. 135) leaves the question open, while Hankel (p. 90), Bretschneider, Tannery, Loria, are inclined to deny Thales a knowledge of similitude of figures. See *Die Geometrie und die Geometer vor Euklides. Ein Historischer Versuch*, von C. A. Bretschneider, Leipzig, 1870, p. 46; *La Géométrie Grecque . . .* par Paul Tannery, Paris, 1887, p. 92; *Le*

The *Eudemian Summary* ascribes to Thales the invention of the theorems on the equality of vertical angles, the equality of the base angles in isosceles triangles, the bisection of the circle by any diameter, and the congruence of two triangles having a side and the two adjacent angles equal respectively. Famous is his application of the last theorem to the determination of the distances of ships from the shore. The theorem that all angles inscribed in a semicircle are right angles is attributed by some ancients to Thales, by others to Pythagoras. Thales thus seems to have originated the geometry of lines and of angles, essentially abstract in character, while the Egyptians dealt primarily with the geometry of surfaces and of solids, empirical in character.[1] It would seem as though the Egyptian priests who cultivated geometry ought at least to have *felt* the truth of the above theorems. We have no doubt that they did, but we incline to the opinion that Thales, like a true philosopher, formulated into theorems and subjected to proof that which others merely felt to be true. If this view is correct, then it follows that Thales in his pyramid and ship measurements was the first to apply theoretical geometry to practical uses.

Thales acquired great celebrity by the prediction of a solar eclipse in 585 B.C. With him begins the study of scientific astronomy. The story goes that once, while viewing the stars, he fell into a ditch. An old woman attending him exclaimed, "How canst thou know what is doing in the heavens, when thou seest not what is at thy feet?"

Astronomers of the Ionic school are Anaximander and Anaximenes. *Anaxagoras*, a pupil of the latter, attempted to square the circle while he was confined in prison. *Approxi-*

Scienze Esatte nell' Antica Grecia, di Gino Loria, in Modena, 1893, Libro I., p. 25.

[1] Allman, p. 16.

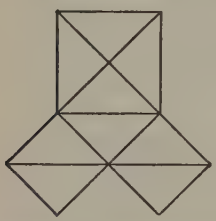
mations to the ratio π occur early among the Egyptians, Babylonians, and Hebrews; but Anaxagoras is the first recorded as attempting the determination of the *exact* ratio — that knotty problem which since his time has been unsuccessfully attacked by thousands. Anaxagoras apparently offered no solution.

II. *The Pythagorean School.* — The life of *Pythagoras* (580 ?–500 ? B.C.) is enveloped in a deep mythical haze. We are reasonably certain, however, that he was born in Samos, studied in Egypt, and, later, returned to the place of his nativity. Perhaps he visited Babylon. Failing in his attempt to found a school in Samos, he followed the current of civilization, and settled at Croton in South Italy (Magna Græcia). There he founded the Pythagorean brotherhood with observances approaching Masonic peculiarity. Its members were forbidden to divulge the discoveries and doctrines of their school. Hence it is now impossible to tell to whom individually the various Pythagorean discoveries must be ascribed. It was the custom among the Pythagoreans to refer every discovery to the great founder of the sect. At first the school flourished, but later it became an object of suspicion on account of its mystic observances. A political party in Lower Italy destroyed the buildings; Pythagoras fled, but was killed at Metapontum. Though politics broke up the Pythagorean fraternity, the school continued to exist for at least two centuries.

Like Thales, Pythagoras wrote no mathematical treatises. The *Eudemian Summary* says: "Pythagoras changed the study of geometry into the form of a liberal education, for he examined its principles to the bottom, and investigated its theorems in an immaterial and intellectual manner."

To Pythagoras himself is to be ascribed the well-known property of the right triangle. The truth of the theorem for the special case when the sides are 3, 4, and 5, respectively, he may have learned from the Egyptians. We are told that Py-

thagoras was so jubilant over this great discovery that he sacrificed a hecatomb to the muses who inspired him. That this is but a legend seems plain from the fact that the Pythagoreans believed in the transmigration of the soul, and, for that reason, opposed the shedding of blood. In the traditions of the late Neo-Pythagoreans the objection is removed by replacing the bloody sacrifice by that of "an ox made of flour"! The demonstration of the law of three squares, given in Euclid, I., 47, is due to Euclid himself. The proof given by Pythagoras has not been handed down to us. Much ingenuity has been expended in conjectures as to its nature. Bretschneider's surmise, that the Pythagorean proof was substantially the same as the one of Bhaskara (given elsewhere), has been well received by Hankel, Allman, Gow, Loria.[1] Cantor thinks it not improbable that the early proof involved the consideration of special cases, of which the isosceles right triangle, perhaps the first, may have been proved in the manner indicated by the adjoining figure.[2] The four lower triangles together equal the four upper. Divisions of squares by their diagonals in this fashion occur in Plato's *Meno*. Since the time of the Greeks the famous Pythagorean Theorem has received many different demonstrations.[3]



A characteristic point in the *method* of Pythagoras and his school was the combination of geometry and arithmetic; an

[1] See Bretschneider, p. 82; Allman, p. 36; Hankel, p. 98; Gow, p. 155; Loria, I., p. 48.

[2] Cantor, I., 185; see also Gow, p. 155, and Allman, p. 29.

[3] See Joh. Jos. Jgn. Hoffmann, *Der Pythagorische Lehrsatz mit zwey und dreysig theils bekannten, theils neuen Beweisen.* Mainz, 1819. See also Jury Wipper, *Sechsundvierzig Beweise des Pythagoräischen Lehrsatzes.* Aus dem Russischen von F. Graap, Leipzig, 1880. The largest collection of proofs is by F. Yanney and J. A. Calderhead in the *Am. Math. Monthly*, Vols. 3 and 4, 1896 and 1897.

arithmetical fact has its analogue in geometry, and *vice versa*. Thus in connection with the law of three squares Pythagoras devised a rule for finding integral numbers representing the lengths of the sides of right triangles: Choose $2n+1$ as one side, then $\frac{1}{2}[(2n+1)^2-1]=2n^2+2n=$ the other side, and $2n^2+2n+1=$ the hypotenuse. If $n=5$, then the three sides are 11, 60, 61. This rule yields only triangles whose hypotenuse exceeds one of the sides by unity.

Ascribed to the Pythagoreans is one of the greatest mathematical discoveries of antiquity — that of *Irrational Quantities*. The discovery is usually supposed to have grown out of the study of the isosceles right triangle.[1] If each of the equal legs is taken as unity, then the hypotenuse, being equal to $\sqrt{2}$, cannot be exactly represented by any number whatever. We may imagine that other numbers, say 7 or $\frac{1}{3}$, were taken to represent the legs; in these and all other cases experimented upon, no number could be found to exactly measure the length of the hypotenuse. After repeated failures, doubtless, "some rare genius, to whom it is granted, during some happy moments, to soar with eagle's flight above the level of human thinking, — it may have been Pythagoras himself, — grasped the happy thought that this problem cannot be solved."[2] As there was nothing in the shape of any geometrical figure which could suggest to the eye the existence of irrationals, their discovery must have resulted from unaided abstract thought. The Pythagoreans saw in irrationals a symbol of the unspeakable. The one who first divulged their theory is said to have suffered shipwreck in consequence, "for the unspeakable and invisible should always be kept secret."[3]

[1] Allman, p. 42, thinks it more likely that the discovery was owing to the problem — to cut a line in extreme and mean ratio.

[2] Hankel, p. 101.

[3] The same story of death in the sea is told of the Pythagorean Hippasus for divulging the knowledge of the dodecaedron.

The theory of parallel lines enters into the Pythagorean proof of the angle-sum in triangles; a line being drawn parallel to the base. In this mode of proof we observe progress from the special to the general, for according to Geminus the early demonstration (by Thales?) of this theorem embraced three different cases: that of equilateral, that of isosceles, and, finally, that of scalene triangles.[1]

Eudemus says the Pythagoreans invented the problems concerning the application of areas, including the cases of defect and excess, as in Euclid, VI., 28, 29. They could also construct a polygon equal in area to a given polygon and similar to another. In a general way it may be said that the Pythagorean plane geometry, like the Egyptian, was much concerned with areas. Conspicuous is the absence of theorems on the circle.

The Pythagoreans demonstrated also that the plane about a point is completely filled by six equilateral triangles, four squares, or three regular hexagons, so that a plane can be divided into figures of either kind. Related to the study of regular polygons is that of the regular solids. It is here that the Pythagoreans contributed to solid geometry. From the equilateral triangle and the square arise the tetraedron, octaedron, cube, and icosaedron — all four probably known to the Egyptians, certainly the first three. In Pythagorean philosophy these solids represent, respectively, the four elements of the physical world: fire, air, earth, and water. In absence of a fifth element, the subsequent discovery of the dodecaedron was made to represent the universe itself. The legend goes that Hippasus perished at sea because he divulged "the sphere with the twelve pentagons."

Pythagoras used to say that the most beautiful of all solids was the sphere; of all plane figures, the circle.

[1] Consult HANKEL, pp. 95, 96.

With what degree of rigour the Italian school demonstrated their theorems we have no sure way of determining. We are safe, however, in assuming that the progress from empirical to reasoned solutions was slow.

Passing to the later Pythagoreans, we meet the name of Philolaus, who wrote a book on Pythagorean doctrines, which he made known to the world. Lastly, we mention the brilliant *Archytas* of Tarentum (428–347 B.C.), who was the only great geometer in Greece when Plato opened his school. He advanced the theory of proportion and wrote on the duplication of the cube.[1]

III. *The Sophist School.* — The periods of existence of the several Greek mathematical "schools" overlap considerably. Thus, Pythagorean activity continued during the time of the sophists, until the opening of the Platonic school.

After the repulse of the Persians at the battle of Salamis in 480 B.C. and the expulsion of the Phœnicians and pirates from the Ægean Sea, Greek commerce began to flourish. Athens gained great ascendancy, and became the centre toward which scholars gravitated. Pythagoreans flocked thither; Anaxagoras brought to Athens Ionic philosophy. The Pythagorean practice of secrecy ceased to be observed; the spirit of Athenian life demanded publicity.[2] All menial work being performed by slaves, the Athenians were people of leisure. That they might excel in public discussions on philosophic or scientific questions, they must be educated. There arose a demand for teachers, who were called *sophists*, or "wise men." Unlike the early Pythagoreans, the sophists accepted pay for their teaching. They taught principally rhetoric, but also philosophy, mathematics, and astronomy.

The geometry of the circle, neglected by the Pythagoreans, was now taken up. The researches of the sophists centre

[1] Consult ALLMAN, pp. 102–127.

[2] ALLMAN, p. 54.

around the three following famous problems, which were to be constructed with aid only of a ruler and a pair of compasses:

(1) The trisection of any angle or arc;

(2) The duplication of the cube; *i.e.* to construct a cube whose *volume* shall be double that of a given cube;

(3) The squaring of the circle; *i.e.* to construct a square or other rectilinear figure whose area exactly equals the area of a given circle.

Certainly no other problems in mathematics have been studied so assiduously and persistently as these. The best Greek intellect was bent upon them; Arabic learning was applied to them; some of the best mathematicians of the Western Renaissance wrestled with them. Trained minds and untrained minds, wise men and cranks, all endeavoured to conquer these problems which the best brains of preceding ages had tried but failed to solve. At last the fact dawned upon the minds of men that, so long as they limited themselves to the postulates laid down by the Greeks, these problems did not admit of solution. This divination was later confirmed by rigorous proof. The Greeks demanded constructions of these problems by ruler and compasses, but no other instruments. In other words, the figure was to consist only of straight lines and of circles. A construction was not *geometrical* if effected by drawing ellipses, parabolas, hyperbolas, or other higher curves. With aid of such curves the Greeks themselves resolved all three problems; but such solutions were objected to as *mechanical*, whereby "the good of geometry is set aside and destroyed, for we again reduce it to the world of sense, instead of elevating and imbuing it with the eternal and incorporeal images of thought, even as it is employed by God, for which reason He always is God" (Plato). Why should the Greeks have admitted the circle into geometrical constructions, but rejected the ellipse, parabola, and the hyperbola — curves of the same order as the circle?

We answer in the words of Sir Isaac Newton:[1] "It is not the simplicity of the equation, but the easiness of the description, which is to determine the choice of our lines for the constructions of problems. For the equation that expresses a parabola is more simple than that that expresses a circle, and yet the circle, by its more simple construction, is admitted before it."[2]

The bisection of an angle is one of the easiest of geometrical constructions. Early investigators, as also beginners in our elementary classes, doubtless expected the division of an angle into three equal parts to be fully as easy. In the special case of a right angle the construction is readily found, but the general case offers insuperable difficulties. One Hippias, very probably *Hippias of Elis* (born about 460 B.C.), was among the earliest to study this problem. Failing to find a construction involving merely circles and straight lines, he discovered a transcendental curve (*i.e.* one which cannot be represented by an algebraic equation) by which an angle could be divided not only into three, but into any number of equal parts. As this same curve was used later in the quadrature of the circle, it received the name of *quadratrix*.[3]

[1] Isaac Newton, *Universal Arithmetick*. Translated by the late Mr. Ralphson; revised by Mr. Cunn. London, 1769, p. 468.

[2] In one of De Morgan's letters to Sir W. R. Hamilton occurs the following: "But what distinguishes the straight line and circle more than anything else, and properly separates them for the purpose of elementary geometry? Their self-similarity. Every inch off a straight line coincides with every other inch, and off a circle with every other off the same circle. Where, then, did Euclid fail? In not introducing the third curve, which has the same property — the *screw*. The right line, the circle, the screw — the representatives of translation, rotation, and the two combined — ought to have been the instruments of geometry. With a screw we should never have heard of the impossibility of trisecting an angle, squaring the circle," etc. — Graves, *Life of Sir William Rowan Hamilton*, 1889, Vol. III., p. 343. However, if Newton's test of easiness of description be applied, then the screw must be excluded.

[3] For a description of the *quadratrix*, see Gow, p. 164. That the in-

The problem "to double the cube" perhaps suggested itself to geometers as an extension to three dimensions of the problem in plane geometry, to double a square. If upon the diagonal of a square a new square is constructed, the area of this new square is exactly twice the area of the first square. This is at once evident from the Pythagorean Theorem. But the construction of a cube double in volume to a given cube brought to light unlooked-for difficulties. A different origin is assigned to this problem by Eratosthenes. The Delians were once suffering from a pestilence, and were ordered by the oracle to double a certain cubical altar. Thoughtless workmen simply constructed a cube with edges twice as long; but brainless work like that did not pacify the gods. The error being discovered, Plato was consulted on this "Delian problem." Eratosthenes tells us a second story: King Minos is represented by an old tragic poet as wishing to erect a tomb for his son; being dissatisfied with the dimensions proposed by the architect, the king exclaimed: "Double it, but fail not in the cubical form." If we trust these stories, then the problem originated in an architectural difficulty.[1] *Hippocrates of Chios* (about 430 B.C.) was the first to show that this problem can be reduced to that of finding between a given line and another twice as long, two mean proportionals, *i.e.* of inserting two lengths between the lines, so that the four shall be in geometrical progression. In modern notation, if a and $2a$ are the two lines, and x and y the mean proportionals, we have the progression $a, x, y, 2a$, which gives $\frac{a}{x} = \frac{x}{y} = \frac{y}{2a}$; whence $x^2 = ay$, $y^2 = 2ax$. Then $x^4 = a^2y^2 = 2a^3x$, $x^3 = 2a^3$. But Hippocrates

ventor of the quadratrix was Hippias of Elis is denied by HANKEL, p. 151, and ALLMAN, p. 94; but affirmed by CANTOR, I., p. 181; BRETSCHNEIDER, p. 94; GOW, p. 163; LORIA, I., p. 66; TANNERY, pp. 108, 131.

[1] GOW, p. 162.

naturally failed to find x, the side of the double cube, by *geometrical* construction. However, the reduction of the problem in solid geometry to one of plane geometry was in itself no mean achievement. He became celebrated also for his success in squaring a lune. This result he attempted to apply to the squaring of the circle.[1] In his study of the Delian and the quadrature problems Hippocrates contributed much to the geometry of the circle. The subject of similar figures, involving the theory of proportion, also engaged his attention. He wrote a geometrical text-book, called the *Elements* (now lost), whereby, no doubt, he contributed vastly towards the progress of geometry by making it more easily accessible to students.

Hippocrates is said to have once lost all his property. Some accounts say he fell into the hands of pirates, others attribute the loss to his own want of tact. Says Aristotle: "It is well known that persons stupid in one respect are by no means so in others. There is nothing strange in this: so Hippocrates, though skilled in geometry, appears to have been in other respects weak and stupid; and he lost, as they say, through his simplicity, a large sum of money by the fraud of the collectors of customs at Byzantium." [2]

A considerable step in advance was the introduction of the *process of exhaustion* by the sophist *Antiphon*, a contemporary of Hippocrates. By inscribing in a circle a square and on its sides erecting isosceles triangles with their vertices in the circumference, and on the sides of these triangles erecting new isosceles triangles, etc., he obtained a succession of regular inscribed polygons of 8, 16, 32, sides, and so on, each polygon approaching in area closer to that of the circle than did the preceding polygon, until the circle was finally *exhausted*. Anti-

[1] For details see Gow, pp. 165–168.

[2] Translated by Allman, p. 57, from *Arist. Eth. ad Eud.*, VII., c. XIV., p. 1247 *a*, 15, ed. Bekker.

phon concluded that a polygon could be thus inscribed, the sides of which, on account of their minuteness, would coincide with the circumference of the circle. Since squares can be found exactly equal in area to any given polygon, there can be constructed a square exactly equal in area to the last polygon inscribed, and therefore equal to the circle itself. Thus it appears that he claimed to have established the possibility of the exact quadrature of the circle. One of his contemporaries, *Bryson of Heraclea*, modified this process of exhaustion by not only inscribing, but also, at the same time, circumscribing regular polygons. He did not claim to secure coincidence between the polygons and circle, but he committed a gross error by assuming the area of the circle to be the exact arithmetical mean of the two polygonal areas.

Antiphon's attempted quadrature involved a point eagerly discussed by philosophers of that time. All other Greek geometers, so far as we know, denied the possibility of the coincidence of a polygon and a circle, for a straight line can never coincide with a circumference or part of it. If a polygon could coincide with a circle, then, says Simplicius, we would have to put aside the notion that magnitudes are divisible *ad infinitum*. We have here a difficult philosophical question, the discussion of which at Athens appears to have greatly influenced and modified Greek mathematical thought in respect to method. The Eleatic philosophical school, with the great dialectician Zeno at its head, argued with admirable ingenuity against the infinite divisibility of a line, or other magnitude. Zeno's position was practically taken by Antiphon in his assumption that straight and curved lines are ultimately reducible to the same indivisible elements.[1] Zeno reasoned by *reductio ad absurdum* against the theory of the infinite divisi-

[1] Allman, p. 56.

bility of a line. He argued that if this theory is assumed to be correct, Achilles could not catch a tortoise. For while he ran to the place where the tortoise had been when he started, the tortoise crept some distance ahead, and while Achilles hastened to that second spot, it again moved forward a little, and so on. Being thus obliged first to reach every one of the infinitely many places which the tortoise had previously occupied, Achilles could never overtake the tortoise. But, as a matter of fact, Achilles could catch a tortoise, therefore, it is wrong to assume that the distance can be divided into an indefinite number of parts. In like manner, "the flying arrow is always at rest; for it is at each moment only in one place." These paradoxes involving the infinite divisibility, and therefore the infinite multiplicity of parts no doubt greatly perplexed the mathematicians of the time. Desirous of constructing an unassailable geometric structure, they banished from their science the ideas of the infinitely little and the infinitely great. Moreover, to meet other objections of dialecticians, theorems evident to the senses (for instance, that two intersecting circles cannot have a common centre) were subjected to rigorous demonstration. Thus the influence of dialecticians like Zeno, themselves not mathematicians, greatly modified geometric science in the direction of increased rigour.[1]

The process of exhaustion, adopted by Antiphon and Bryson, was developed into the perfectly rigorous *method of exhaustion.* In finding, for example, the ratio between the areas of two circles, similar polygons were inscribed and by increasing the number of sides, the spaces between the polygons and circles nearly exhausted. Since the polygonal areas were to each other as the squares of the diameters, geometers doubtless divined the theorem attributed to Hippocrates of Chios, that

[1] Consult further, HANKEL, p. 118; CANTOR, I., p. 198; ALLMAN, p. 55; LORIA, I., p. 53.

the circles themselves are to each other as the square of their diameters. But in order to exclude all vagueness or possibility of doubt, later Greek geometers applied reasoning like that in Euclid, XII., 2, which we give in condensed form as follows: Let C, c be two circular areas; D, d, the diameters. Then, if the proportion $D^2 : d^2 = C : c$ is not true, suppose that $D^2 : d^2 = C : c'$. If $c' < c$, then a polygon p can be inscribed in the circle c which comes nearer to c in area than does c'. If P be the corresponding polygon in C, then $P : p = D^2 : d^2 = C : c'$, and $P : C = p : c'$. Since $p > c'$, we have $P > C$, which is absurd. Similarly, c' cannot exceed c. As c' cannot be larger, nor smaller than c, it must be equal to c. Q.E.D. Here we have exemplified the *method of exhaustion*, involving the process of reasoning, designated the *reductio ad absurdum*. Hankel refers this method of exhaustion back to Hippocrates of Chios, but the reasons for assigning it to this early writer, rather than to Eudoxus, seem insufficient.[1]

IV. *The Platonic School.* — After the Peloponnesian War (431–404 B.C.), the political power of Athens declined, but her leadership in philosophy, literature, and science became all the stronger. She brought forth such men as *Plato* (429 ?–347 B.C.), the strength of whose mind has influenced philosophical thought of all ages. Socrates, his early teacher, despised mathematics. But after the death of Socrates, Plato travelled extensively and came in contact with several prominent mathematicians. At Cyrene he studied geometry with Theodorus; in Italy he met the Pythagoreans. Archytas of Tarentum and Timæus of Locri became his intimate friends. About 389 B.C. Plato returned to Athens, founded a school in the groves of the *Academia*, and devoted the remainder of his life to teaching and writing. Unlike his master, Socrates, Plato placed

[1] Consult HANKEL, p. 122; GOW, p. 173; CANTOR, I., pp. 242, 247.

great value upon the mind-developing power of mathematics. "Let no one who is unacquainted with geometry enter here," was inscribed over the entrance to his school. Likewise Xenocrates, a successor of Plato, as teacher in the Academy, declined to admit a pupil without mathematical training, "Depart, for thou hast not the grip of philosophy." The *Eudemian Summary* says of Plato that "he filled his writings with mathematical terms and illustrations, and exhibited on every occasion the remarkable connection between mathematics and philosophy."

Plato was not a professed mathematician. He did little or no original work, but he encouraged mathematical study and suggested improvements in the logic and methods employed in geometry. He turned the instinctive logic of the earlier geometers into a method to be used consciously and without misgiving.[1] With him begin careful definitions and the consideration of postulates and axioms. The Pythagorean definition, "a point is unity in position," embodying a philosophical theory, was rejected by the Platonists; a point was defined as "the beginning of a straight line" or "an indivisible line." According to Aristotle the following definitions were also current: The point, the line, the surface, are respectively the boundaries of the line, the surface, and the solid; a solid is that which has three dimensions. Aristotle quotes from the Platonists the axiom: If equals be taken from equals, the remainders are equal. How many of the definitions and axioms were due to Plato himself we cannot tell. Proclus and Diogenes Laertius name Plato as the inventor of the method of proof called *analysis*. To be sure, this method had been used unconsciously by Hippocrates and others, but it is generally believed that Plato was the one who turned the un-

[1] Gow, pp. 175, 176. See also Hankel, pp. 127–150.

conscious logic into a conscious, legitimate method. The development and perfection of this method was certainly a great achievement, but Allman (p. 125) is more inclined to ascribe it to Archytas than to Plato.

The terms *synthesis* and *analysis* in Greek mathematics had a different meaning from what they have in modern mathematics or in logic.[1] The oldest definition of analysis as opposed to synthesis is given in Euclid, XIII., 5, which was most likely framed by Eudoxus:[2] "Analysis is the obtaining of the thing sought by assuming it and so reasoning up to an admitted truth; synthesis is the obtaining of the thing sought by reasoning up to the inference and proof of it."[3]

Plato gave a powerful stimulus to the study of solid geometry. The sphere and the regular solids had been studied somewhat by the Pythagoreans and Egyptians. The latter,

[1] HANKEL, pp. 137–150; also HANKEL, *Mathematik in den letzten Jahrhunderten*, Tübingen, 1884, p. 12.

[2] BRETSCHNEIDER, p. 168.

[3] Greek mathematics exhibits different types of analysis. One is the *reductio ad absurdum*, in the *method of exhaustion*. Suppose we wish to prove that "A is B." We assume that A is *not* B; then we form a synthetic series of conclusions: *not* B is C, C is D, D is E; if now A is *not* E, then it is impossible that A is *not* B; *i.e.*, A is B. Q.E.D. Verify this process by taking Euclid, XII., 2, given above. Allied to this is the *theoretic* analysis: To *prove* that A is B *assume* that A is B, then B is C, C is D, D is E, E is F; hence A is F. If this last is known to be false, then A is *not* B; if it is known to be true, then the reasoning thus far is not conclusive. To remove doubt we must follow the reverse process, A is F, F is E, E is D, D is C, C is B; therefore A is B. This second case involves two processes, the analytic followed by the synthetic. The only aim of the analytic is to aid in the discovery of the synthetic. Of greater importance to the Greeks was the *problematic* analysis, applied in constructions intended to satisfy given conditions. The construction is *assumed* as accomplished; then the geometric relations are studied with the view of discovering a synthetic solution of the problem. For examples of proofs by analysis, consult HANKEL, p. 143; GOW, p. 178; ALLMAN, pp. 160–163; TODHUNTER'S *Euclid*, 1869, Appendix, pp. 320–328.

of course, were more or less familiar with the geometry of the pyramid. In the Platonic school, the prism, pyramid, cylinder, and cone were investigated. The study of the cone led Menæchmus to the discovery of the conic sections. Perhaps the most brilliant mathematician of this period was *Eudoxus*. He was born at Cnidus about 408 B.C., studied under Archytas and, for two months, under Plato. Later he taught at Cyzicus. At one time he visited, with his pupils, the Platonic school. He died at Cyzicus in 355 B.C. Among the pupils of Eudoxus at Cyzicus who afterwards entered the academy of Plato were Menæchmus, Dinostratus, Athenæus, and Helicon. The fame of the Academy is largely due to these. The *Eudemian Summary* says that Eudoxus "first increased the number of general theorems, added to the three proportions three more, and raised to a considerable quantity the learning, begun by Plato, on the subject of the section, to which he applied the analytic method." By this "section" is meant, no doubt, the "golden section," which cuts a line in extreme and mean ratio. He proved, says Archimedes, that a pyramid is exactly one-third of a prism, and a cone one-third of a cylinder, having equal base and altitude. That spheres are to each other as the cubes of their radii, was probably established by him. The *method of exhaustion* was used by him extensively and was probably his own invention.[1]

[1] The *Eudemian Summary* mentions, besides the geometers already named, Theætetus of Athens, to whom Euclid is supposed to be indebted in the composition of the 10th book, treating of incommensurables (ALLMAN, pp. 206–215); Leodamas of Thasos; Neocleides and his pupil Leon, who wrote a geometry; Theudius of Magnesia, who also wrote a geometry or *Elements;* Hermotimus of Colophon, who discovered many propositions in Euclid's *Elements;* Amyclas of Heraclea, Cyzicenus of Athens, and Philippus of Mende. The pre-Euclidean text-books just mentioned are not extant.

V. *The First Alexandrian School.* — During the sixty-six years following the Peloponnesian War — a period of political decline — Athens brought forth some of the greatest and most subtle thinkers of Greek antiquity. In 338 B.C. she was conquered by Philip of Macedon and her power broken forever. Soon after, Alexandria was founded by Alexander the Great, and it was in this city that literature, philosophy, science, and art found a new home.

In the course of our narrative, we have seen geometry take feeble root in Egypt; we have seen it transplanted to the Ionian Isles; thence to Lower Italy and to Athens; now, at last, grown to substantial and graceful proportions, we see it transferred to the land of its origin, and there, newly invigorated, expand in exuberant growth.

Perhaps the founder, certainly a central figure, of the Alexandrian mathematical school was *Euclid* (about 300 B.C.). No ancient writer in any branch of knowledge has held such a commanding position in modern education as has Euclid in elementary geometry. "The sacred writings excepted, no Greek has been so much read or so variously translated as Euclid."[1]

After mentioning Eudoxus, Theætetus, and other members of the Platonic school, Proclus[2] adds the following to the *Eudemian Summary:*

"Not much later than these is Euclid, who wrote the *Elements*, arranged much of Eudoxus's work, completed much of Theætetus's, and brought to irrefragable proof propositions which had been less strictly proved by his predecessors. Euclid lived during the reign of the first Ptolemy, for he is quoted by Archimedes in his first book; and it is said, moreover, that Ptolemy once asked him, whether in geometric mat-

[1] DE MORGAN, "Eucleides" in *Smith's Dictionary of Greek and Roman Biography and Mythology.* We commend this remarkable article to all.

[2] PROCLUS (Ed. FRIEDLEIN), p. 68.

ters there was not a shorter path than through his *Elements*, to which he replied that there was no royal road to geometry.[1] He is, therefore, younger than the pupils of Plato, but older than Eratosthenes and Archimedes, for these are contemporaries, as Eratosthenes informs us. He belonged to the Platonic sect and was familiar with Platonic philosophy, so much so, in fact, that he set forth the final aim of his work on the *Elements* to be the construction of the so-called Platonic figures (regular solids)."[2] Pleasing are the remarks of Pappus,[3] who says he was gentle and amiable to all those who could in the least degree advance mathematical science. Stobæus[4] tells the following story: "A youth who had begun to read geometry with Euclid, when he had learned the first proposition, inquired, 'What do I get by learning those things?' So Euclid called his slave and said, 'Give him threepence, since he must gain out of what he learns.'"

Not much more than what is given in these extracts do we know concerning the life of Euclid. All other statements about him are trivial, of doubtful authority, or clearly erroneous.[5]

[1] "This piece of wit has had many imitators. 'Quel diable,' said a French nobleman to Rohault, his teacher in geometry, 'pourrait entendre cela?' to which the answer was, 'Ce serait un diable qui aurait de la patience.' A story similar to that of Euclid is related by Seneca (Ep. 91, cited by August) of Alexander." De Morgan, *op. cit.*

[2] This statement of Euclid's aim is obviously erroneous.

[3] Pappus (Ed. Hultsch), pp. 676–678.

[4] Quoted by Gow, p. 195 from *Floril.* IV., p. 250.

[5] Syrian and Arabian writers claim to possess more information about Euclid; they say that his father was Naucrates, that Euclid was a Greek born in Tyre, that he lived in Damascus and edited the *Elements* of Apollonius. For extracts from Arabic authors and for a list of books on the principal editions of Euclid, see Loria, II., pp. 10, 11, 17, 18. During the middle ages the geometer Euclid was confused with Euclid of Megara, a pupil of Socrates. Of interest is the following quotation from De Mor

Though the author of several works on mathematics and physics, the fame of Euclid has at all times rested mainly upon his book on geometry, called the *Elements*. This book was so far superior to the *Elements* written by Hippocrates, Leon, and Theudius, that the latter works soon perished in the struggle for existence. The great rôle that it has played in geometric teaching during all subsequent centuries, as also its strong and weak points, viewed in the light of pedagogical science and of modern geometrical discoveries, will be discussed more fully later. At present we confine ourselves to a brief critical account of its contents.

Exactly how much of the *Elements* is original with Euclid, we have no means of ascertaining. Positive we are that certain early editors of the *Elements* were wrong in their view that a finished and unassailable system of geometry sprang at once from the brain of Euclid, "an armed Minerva from the head of Jupiter." Historical research has shown that Euclid got the larger part of his material from the eminent mathematicians who preceded him. In fact, the proof of the "Theorem of Pythagoras" is the only one directly ascribed to him. Allman[1] conjectures that the substance of Books I., II., IV. comes from the Pythagoreans, that the substance of Book VI. is due to the Pythagoreans and Eudoxus, the latter contributing the doctrine of proportion as applicable to incommensurables and also the Method of Exhaustions (Book XII.), that Theætetus contributed much toward Books X. and XIII., that the principal part of the original work of Euclid himself is to

GAN, *op. cit.*: "In the frontispiece to Whiston's translation of Tacquet's Euclid there is a bust, which is said to be taken from a brass coin in possession of Christina of Sweden; but no such coin appears in the published collection of those in the cabinet of the queen of Sweden. Sidonius Apollinaris says (*Epist.* XI., 9) that it was the custom to paint Euclid with the fingers extended (laxatis), as if in the act of measurement."

[1] ALLMAN, pp. 211, 212.

be found in Book X. The greatest achievement of Euclid, no doubt, was the co-ordinating and systematizing of the material handed down to him. He deserves to be ranked as one of the greatest systematizers of all time.

The contents of the *Elements* may be briefly indicated as follows: Books I., II., III., IV., VI. treat of plane geometry; Book V., of the theory of proportion applicable to magnitudes in general; Books VII., VIII., IX., of arithmetic; Book X., of the arithmetical characteristics of divisions of straight lines (*i.e.* of irrationals); Books XI., XII., of solid geometry; Books XIII., XIV., XV., of the regular solids. The last two books are apocryphal, and are supposed to have been written by Hypsicles and Damascius, respectively.[1]

Difference of opinion long existed regarding the merits of the *Elements* as a scientific treatise. Some regarded it as a work whose logic is in every detail perfect and unassailable, while others pronounced it to be "riddled with fallacies."[2] In our opinion, neither view is correct. That the text of the *Elements* is not free from faults is evident to any one who reads the commentators on Euclid. Perhaps no one ever surpassed Robert Simson in admiration for the great Alexandrian. Yet Simson's "notes" to the text disclose numerous defects. While Simson is certainly wrong in attributing to blundering editors *all* the defects which he noticed in the *Elements*, early editors are doubtless responsible for many of them. Most of the emendations pertain to points of minor importance. The work as a whole possesses a high standard of accuracy. A minute examination of the text has disclosed to commentators

[1] See Gow, p. 272; Cantor, I., pp. 358, 501. Book XV., in the opinion of Heiberg and others, consists of three parts, of which the third is perhaps due to Damascius. See Loria, II., pp. 88–92.

[2] See C. S. Peirce in *Nation*, Vol. 54, 1892, pp. 116, 366, and in the *Monist*, July, 1892, p. 539; G. B. Halsted, *Educational Review*, 8, 1894, pp. 91–93.

an occasional lack of extreme precision in the statement of what is assumed without proof, for truths are treated as self-evident which are not found in the list of postulates.[1] Again, Euclid sometimes assumes what might be proved; as when in the very definitions he asserts that the diameter of a circle bisects the figure,[2] which might be readily proved from the axioms. He defines a plane angle as the "inclination of two straight lines to one another, which meet together, but are not in the same straight line," but leaves the idea of angle magnitude somewhat indefinite by his failure to give a test for equality of two angles or to state what constitutes the sum or difference of two angles.[3] Sometimes Euclid fails to consider or give all the special cases necessary for the full and complete proof of a theorem.[4] Such instances of defects which friendly critics have found in the *Elements* show that Euclid is not infallible.[5] But in noticing these faults, we must not lose

[1] For instance, the intersection of the circles in I., 1, and in I., 22. See also H. M. Taylor's *Euclid*, 1893, p. vii.

[2] De Morgan, article "Euclid of Alexandria," in the *English Cyclopædia*.

[3] See Simon Newcomb, *Elements of Geometry*, 1884, Preface; H. M. Taylor's *Euclid*, p. 8; De Morgan, *The Connexion of Number and Magnitude*, London, 1836, p. 85.

[4] Consult Todhunter's *Euclid*, notes on I., 35; III., 21; XI., 21. Simson's *Euclid*, notes on I., 7; III., 35.

[5] A proof which has been repeatedly attacked is that of I., 16, which "uses no premises not as true in the case of spherical as in that of plane triangles; and yet the conclusion drawn from these premises is known to be false of spherical triangles." See *Nation*, Vol. 54, pp. 116, 366; E. T. Dixon, in *Association for the Improvement of Geometrical Teaching* (A. I. G. T.), 17th General Report, 1891, p. 29; Engel und Stäckel, *Die Theorie der Parallellinien von Euklid bis auf Gauss*, Leipzig, 1895, p. 11, note. (Hereafter we shall cite this book as Engel and Stäckel.) In the *Monist*, July, 1894, p. 485, G. B. Halsted defends the proof of I., 16, arguing that spherics are ruled out by the postulate, "Two straight lines cannot enclose a space." If we were sure that Euclid used this postulate, then the proof of I., 16 would be unobjectionable, but it is probable that

sight of the general excellence of the work as a scientific treatise — an excellence which in 1877 was fittingly recognized by a committee of the British Association for the Advancement of Science (comprising some of England's ablest mathematicians) in their report that "no text-book that has yet been produced is fit to succeed Euclid in the position of authority."[1] As already remarked, some editors of the *Elements*, particularly Robert Simson, wrote on the supposition that the original Euclid was perfect, and any defects in the text as known to them, they attributed to corruptions. For example, Simson thinks there should be a definition of compound ratio at the beginning of the fifth book; so he inserts one and assures us that it is the very definition given by Euclid. Not a single manuscript, however, supports him. The text of the *Elements* now commonly used is Theon's. Simson was inclined to make him the scapegoat for all defects which he thought he discovered in Euclid. But a copy of the *Elements* sent with other manuscripts from the Vatican to Paris by Napoleon I. is believed to be anterior to Theon's recension; this differs but slightly from Theon's version, showing that the faults are probably Euclid's own.

At the beginning of our modern translations of the *Elements* (Robert Simson's or Todhunter's, for example), under the head of definitions are given the assumptions of such notions as the point, line, etc., and some verbal explanations. Then follow three postulates or demands, (1) that a line may be drawn from any point to any other, (2) that a line may be indefinitely produced, (3) that a circle may be drawn with any radius and any point as centre. After these come twelve axioms.[2] The term

the postulate was omitted by Euclid and supplied by some commentator. See Heiberg, *Euclidis Elementa*.

[1] A. I. G. T., 6th *General Report*, 1878, p. 14.

[2] Of these twelve "axioms," five are supposed not to have been given by

axiom was used by Proclus, but not by Euclid. He speaks instead of "common notions" — common either to all men or to all sciences. The first nine "axioms" relate to all kinds of magnitudes (things equal to the same thing are equal to each other, etc., the whole is greater than its part),[1] while the last three (two straight lines cannot enclose a space; all right angles are equal to one another; the parallel-axiom) relate to space only. While in nearly all but the most recent of modern editions of Euclid the geometric "axioms" are placed in the same category with the other nine, it is certainly true that Euclid sharply distinguished between the two classes. An immense preponderance of manuscripts places the "axioms" relating to space among the postulates.[2] This is their proper place, for modern research has shown that they are *assumptions* and not *common notions* or axioms. It is not known who first made the unfortunate change. In this respect there should

Euclid, viz. the four on inequalities and the one, "two straight lines cannot enclose a space." Thus HEIBERG and MENGE, in their Latin and Greek edition of 1883, omit all five. See also ENGEL and STÄCKEL, p. 8, note.

[1] "That the whole is greater than its part is not an axiom, as that eminently bad reasoner, Euclid, made it to be.... Of finite collections it is true, of infinite collections false." — C. S. PEIRCE, *Monist*, July, 1892, p. 539. Peirce gives illustrations in which, for infinite collections, the "axiom" is untrue. Nevertheless, we are unwilling to admit that the assuming of this axiom proves Euclid a bad reasoner. Euclid had nothing whatever to do with infinite collections. As for finite collections, it would seem that nothing can be *more* axiomatic. To infinite collections the terms *great* and *small* are inapplicable. On this point see GEORG CANTOR, "Ueber eine Eigenschaft des Inbegriffes reeller algebraischer Zahlen," *Crelle Journal*, 77, 1873; or for a more elementary discussion, see FELIX KLEIN, *Ausgewählte Fragen der Elementargeometrie*, ausgearbeitet von F. TÄGERT, Leipzig, 1895, p. 39. [This work will be cited hereafter as KLEIN.] That "the whole is greater than its part" does not apply in the comparison of infinites was recognized by Bolyai. See HALSTED'S BOLYAI'S *Science Absolute of Space*, 4th Ed., 1896, § 24, p. 20.

[2] HANKEL, *Die Complexen Zahlen*, Leipzig, 1867, p. 52.

be a speedy return to Euclid's practice. The parallel-postulate plays a very important rôle in the history of geometry.[1] Most modern authorities hold that Euclid missed one of the postulates, that of rigidity (or else that of equal variation), which demands that figures may be moved about in space without any alteration in form or magnitude (or that all moving figures change equally and each fills the same space when brought back to its original position).[2] The rigidity postulate is given in recent geometries, but a contradictory of it, stated above (the postulate of equal variation) would likewise admit figures to be compared by the method of superposition, and "everything would go on quite as well" (Clifford). Of the two, the former is the simpler postulate and is, moreover, in accordance with what we conceive to be our every-day experience. G. B. Halsted contends that Euclid did *not* miss the rigidity assumption and is justified in not making it, since he covers it by his assumption 8: "Magnitudes which can be made to coincide with one another are equal to one another."

In the first book, Euclid only once imagines figures to be moved relatively to each other, namely, in proving proposi-

[1] The parallel-postulate is: "If a straight line meet two straight lines, so as to make the two interior angles on the same side of it taken together less than two right angles, these straight lines, being continually produced, shall at length meet on that side on which are the angles which are less than two right angles." In the various editions of Euclid, different numbers are assigned to the "axioms." Thus, the parallel-postulate is in old manuscripts the 5th *postulate*. This place is also assigned to it by F. Peyrard (who was the first to critically compare the various MSS.) in his edition of Euclid in French and Latin, 1814, and by Heiberg and Menge in their excellent annotated edition of Euclid's works, in Greek and Latin, Leipzig, 1883. Clavius calls it the 13th *axiom;* Robert Simson, the 12th *axiom;* others (Bolyai for instance), the 11th *axiom*.

[2] Consult W. K. Clifford, *The Common Sense of the Exact Sciences*, 1885, p. 54. "(1) Different things changed equally, and (2) anything which was carried about and brought back to its original position filled the same space."

tion 4: two triangles are equal if two sides and the included angle are equal respectively. To bring the triangles into coincidence, one triangle may have to be turned over, but Euclid is silent on this point. "Can it have escaped his notice that in plane geometry there is an essential difference between motion of translation and reversion?"[1]

Book V., on proportions of magnitudes, has been greatly admired because of its rigour of treatment.[2] Beginners find the book difficult. It has been the chief battle-ground of discussion regarding the fitness of the *Elements* as a text-book for beginners.

Book X. (as also Books VII., VIII., IX., XIII., XIV., XV.) is omitted from modern school editions. But it is the most wonderful book of all. Euclid investigates every possible variety of lines which can be represented by $\sqrt{\sqrt{a} \pm \sqrt{b}}$, a and b representing two commensurable lines, and obtains 25 species. Every individual of every species is incommensurable with all the individuals of every other species. De Morgan was enthusiastic in his admiration of this book.[3]

[1] ENGEL and STÄCKEL, p. 8, note.

[2] For interesting comments on Books V. and VI., see HANKEL, pp. 389–404.

[3] See his articles "Eucleides" in SMITH's *Dic. of Greek and Roman Biog. and Myth.* and "Irrational Quantity" in the *Penny Cyclopædia* or in the *English Cyclopædia.* See also NESSELMANN, pp. 165–183. In connection with this subject of irrationals a remark by Dedekind is of interest. See RICHARD DEDEKIND, *Was sind und was sollen die Zahlen*, Braunschweig, 1888, pp. xii and xiii. He points out that all Euclid's constructions of figures could be made, even if the plane were not continuous; that is, even if certain points in the plane were imagined to be punched out, so as to give it the appearance of a sieve. All the points in Euclid's constructions would lie between the holes; no point of the constructions would fall into a hole. The explanation of all this is to be sought in the fact that Euclid deals with certain algebraic irrationals, to the exclusion of the transcendental.

The main differences in subject-matter between the *Elements* and our modern school geometries consist in this: the modern works pay less attention to the "Platonic figures," but add theorems on triangles and quadrilaterals inscribed in or circumscribed about a circle, on the centre of gravity of the triangle, on spherical triangles (in general, the geometry of a spherical surface), and, perhaps, on some of the more recent discoveries pertaining to the geometry of the plane triangle and circle.

The main difference as to method, between Euclid and his modern rivals, lies in the treatment of proportion and the development of size-relations. His geometric theory of proportion enables him to study these relations without reference to mensuration. The *Elements* and all Greek geometry before Archimedes eschew mensuration. The theorem that the area of a triangle equals half the product of its base and its altitude, or that the area of a circle equals π times the square of the radius, is foreign to Euclid. In fact he nowhere finds an approximation to the ratio between the circumference and diameter. Another difference is that Euclid, unlike the great majority of modern writers, never draws a line or constructs a figure until he has actually shown the possibility of such construction with aid only of the first three of his postulates or of some previous construction. The first three propositions of Book I. are not theorems, but problems, (1) to describe an equilateral triangle, (2) from a given point to draw a straight line equal to a given straight line, (3) from the greater of two straight lines to cut off a part equal to the less. It is only by the use of hypothetical figures that modern books can relegate all constructions to the end of chapters. For instance, an angle is imagined to be bisected, before the possibility and method of bisecting it has been shown. One of the most startling examples of hypothetical constructions is the division

of a circumference into *any desired* number of equal parts, given in a modern text-book. This appears all the more startling, when we remember that one of the discoveries which make the name of Gauss immortal is the theorem that, besides regular polygons of 2^n, 3, 5 sides (and combinations therefrom), only polygons whose number of sides is a prime number larger than five, and of the form $p = 2^{2^n} + 1$, can be inscribed in a circle with aid of Euclid's postulates, *i.e.* with aid of ruler and compasses only.[1] Is the absence of hypothetical constructions commendable? If the aim is rigour, we answer emphatically, *Yes.* If the aim is adherence to recognized pedagogical principles, then we answer, in a general way, that, in the transition from the concrete to the more abstract geometry, it often seems desirable to let facts of observation take the place of abstruse processes of reasoning. Even Euclid resorts to observation for the fact that his two circles in I., 1 cut each other.[2] Reasoning too difficult to be grasped does not develop the mind.[3] Moreover, the beginner, like the ancient Epicureans, takes no interest in trains of reasoning which *prove* to him what he has *long known;* geometrical reasoning is more apt to interest him when it discloses new facts. Thus, pedagogics may reasonably demand *some* concessions from demonstrative rigour.

Of the other works of Euclid we mention only the *Data*, probably intended for students who had completed the *Elements* and wanted drill in solving new problems; a lost work on *Fallacies*, containing exercises in detecting fallacies; a treatise on *Porisms*, also lost, but restored by Robert Simson and Michel Chasles.

[1] Klein, p. 2.

[2] The Epicureans, says Proclus, blamed Euclid for proving some things which were evident without proof. Thus, they derided I., 20 (two sides of a triangle are greater than the third) as being manifest even to asses.

[3] For discussions of the subject of Hypothetical Constructions, see E. L. Richards in *Educat. Review*, Vol. III., 1892, p. 34; G. B. Halsted in same journal, Vol. IV., 1893, p. 152.

The period in which Euclid flourished was the golden era in Greek mathematical history. This era brought forth the two most original mathematicians of antiquity, *Archimedes* and *Apollonius of Perga*. They rank among the greatest mathematicians of all time. Only a small part of their discoveries can be described here.

Archimedes (287 ?–212 B.C.) was born at Syracuse in Sicily. Cicero tells us he was of low birth. He visited Egypt and, perhaps, studied in Alexandria; then returned to his native place, where he made himself useful to his admiring friend and patron, King Hieron, by applying his extraordinary inventive powers to the construction of war-engines, by which he inflicted great loss on the Romans, who, under Marcellus, were besieging the city. That by the use of mirrors reflecting the sun's rays he set on fire the Roman ships, when they came within bow-shot of the walls, is probably a fiction. Syracuse was taken at length by the Romans, and Archimedes died in the indiscriminate slaughter which followed. The story goes that, at the time, he was studying some geometrical diagram drawn in the sand. To an approaching Roman soldier he called out, "Don't spoil my circles," but the soldier, feeling insulted, killed him. The Roman general Marcellus, who admired his genius, raised in his honour a tomb bearing the figure of a sphere inscribed in a cylinder. The Sicilians neglected the memory of Archimedes, for when Cicero visited Syracuse, he found the tomb buried under rubbish.

While admired by his fellow-citizens mainly for his mechanical inventions, he himself prized more highly his discoveries in pure science.

Of special interest to us is his book on the *Measurement of the Circle*.[1] He proves first that the circular area is equal to

[1] A recent standard edition of his works is that of HEIBERG, Leipzig, 1880–81. For a fuller account of his *Measurement of the Circle*, see

that of a right triangle having the length of the circumference for its base and the radius for its altitude. To find this base is the next task. He first finds an upper limit for the ratio of the circumference to the diameter. After constructing an equilateral triangle with its vertex in the centre of the circle and its base tangent to the circle, he bisects the angle at the centre and determines the ratio of the base to the altitude of one of the resulting right triangles, taking the irrational square root a little too small. Next, the central angle of this right triangle is bisected and the ratio of its legs determined. Then the central angle of this last right triangle is bisected and the ratio of its legs computed. This bisecting and computing is carried on four times, the irrational square roots being taken every time a little too small. The ratio of the last two legs considered is $> 4673\frac{1}{2} : 153$. But the shorter of the legs having this ratio is the side of a regular circumscribed polygon. This leads him to the conclusion that the ratio of the circumference to the diameter is $< 3\frac{1}{7}$. Next, he finds a lower limit by inscribing regular polygons of 6, 12, 24, 48, 96 sides, finding for each successive polygon its perimeter. In this way he arrives at the lower limit $3\frac{10}{71}$. Hence the final result, $3\frac{1}{7} > \pi > 3\frac{10}{71}$, an approximation accurate enough for most purposes.

Worth noting is the fact that while approximations to π were made before this time by the Egyptians, no sign of such computations occur in Euclid and his Greek predecessors. Why this strange omission? Perhaps because Greek ideality excluded all calculation from geometry, lest this noble science lose its rigour and be degraded to the level of geodesy or surveying. Aristotle says that truths pertaining to geometrical magnitudes cannot be proved by anything so foreign to geom-

Cantor, I., pp. 300–303, 316, 319; Loria, II., pp. 126–132; Gow, pp. 233–237; H. Weissenborn, *Die Berechnung des Kreis-Umfanges bei Archimedes und Leonardo Pisano*, Berlin, 1894.

etry as arithmetic. The real reason, perhaps, may be found in the contention by some ancient critics that it is not evident that a straight line can be equal in length to a curved line; in particular that a straight line exists which is equal in length to the circumference. This involves a real difficulty in geometrical reasoning. Euclid bases the equality between lines or between areas on congruence. Now, since no curved line, or even a part of a curved line, can be made to *exactly* coincide with a straight line or even a part of a straight line, no comparisons of length between a curved line and a straight line can be made. So, in Euclid, we nowhere find it given that a curved line is equal to a straight line. The method employed in Greek geometry truly excludes such comparisons; according to Duhamel, the more modern idea of a limit is needed to logically establish the possibility of such comparisons.[1] On Euclidean assumptions it cannot even be *proved* that the perimeter of a circumscribed (inscribed) polygon is greater (smaller) than the circumference. Some writers tacitly resort to observation; they can *see* that it is so.

Archimedes went a step further and assumed not only this, but, *trusting to his intuitions*, tacitly made the further assumption that a straight line exists which equals the circumference in length. On this new basis he made a valued contribution to geometry. No doubt we have here an instance of the usual course in scientific progress. Epoch-making discoveries, at their birth, are not usually supported on every side by unyielding logic; on the contrary, intuitive insight guides the seeker over difficult places. As further examples of this we instance the discoveries of Newton in mathematics and of Maxwell in physics. The *complete* chain of reasoning by which the truth of a discovery is established is usually put together at a later period.

[1] G. B. Halsted in *Tr. Texas Academy of Science*, I., p. 96.

Is not the same course of advancement observable in the individual mind? We first arrive at truths without fully grasping the reasons for them. Nor is it always best that the young mind should, from the start, make the effort to grasp them all. In geometrical teaching, where the reasoning is too hard to be mastered, if observation can conveniently assist, accept its results. A student cannot wait until he has mastered limits and the calculus, before accepting the truth that the circumference is greater than the perimeter of an inscribed polygon.

Of all his discoveries Archimedes prized most highly those in his book on the *Sphere and Cylinder*. In this he uses the celebrated statement, "the straight line is the shortest path between two points," but he does not offer this as a formal definition of a straight line.[1] Archimedes proves the new theorems that the surface of a sphere is equal to four times a great circle; that the surface of a segment of a sphere is equal to a circle whose radius is the straight line drawn from the vertex of the segment to the circumference of its basal circle; that the volume and the surface of a sphere are $\frac{2}{3}$ of the volume and surface, respectively, of the cylinder circumscribed about the sphere. The wish of Archimedes, that the figure for the last proposition be inscribed upon his tomb, was carried out by the Roman general Marcellus.

Archimedes further advanced solid geometry by adding to the five "Platonic figures," thirteen *semi-regular solids*, each bounded by regular polygons, but not all of the same kind. To elementary geometry belong also his fifteen *Lemmas*.[2]

Of the "Great Geometer," *Apollonius of Perga*, who flourished about forty years after Archimedes, and investigated the properties of the conic sections, we mention, besides his

[1] Cantor, I., 298.

[2] Consult Gow, p. 232; Cantor, I., p. 298.

celebrated work on *Conics*, only a lost work on *Contacts*, which Vieta and others attempted to restore from certain lemmas given by Pappus. It contained the solution of the celebrated "Apollonian Problem": Given three circles, to find a fourth which shall touch the three. Even in modern times this problem has given stimulus toward perfecting geometric methods.[1]

With Euclid, Archimedes, and Apollonius, the eras of Greek geometric discovery reach their culmination. But little is known of the history of geometry from the time of Apollonius to the beginning of the Christian era. In this interval falls *Zenodorus*, who wrote on *Figures of Equal Periphery*. This book is lost, but fourteen propositions of it are preserved by Pappus and also by Theon. Here are three of them: "The circle has a greater area than any polygon of equal periphery," "Of polygons of the same number of sides and of equal periphery the regular is the greatest," "Of all solids having surfaces equal in area, the sphere has the greatest volume."

Between 200 and 100 B.C. lived *Hypsicles*, the supposed author of the fourteenth book in Euclid's *Elements*. His treatise on *Risings* is the earliest Greek work giving the division of the circle into 360 degrees after the manner of the Babylonians.

Hipparchus of Nicæa in Bithynia, the author of the famous theory of epicycles and eccentrics, is the greatest astronomer of antiquity. Theon of Alexandria tells us that he originated the science of *trigonometry* and calculated a "table of chords" in twelve books (not extant). Hipparchus took astronomical observations between 161 and 126 B.C.

A writer whose tone is very different from that of the great writers of the First Alexandrian School was *Heron of Alexan-*

[1] Consult E. Schilke, *Die Lösungen und Erweiterungen des Apollonischen Berührungsproblems*, Berlin, 1880.

dria, also called Heron the Elder.[1] As he was a practical surveyor, it is not surprising to find little resemblance between his writings and those of Euclid or Apollonius.

Heron was a pupil of Ctesibius, who was celebrated for his mechanical inventions, such as the hydraulic organ, water-clock, and catapult. It is believed by some that Heron was a son of Ctesibius. Heron's invention of the eolipile and a curious mechanism, known as "Heron's Fountain," display talent of the same order as that of his master. Great uncertainty exists regarding his writings. Most authorities believe him to be the author of a work, entitled *Dioptra*, of which three quite dissimilar manuscripts are extant. Marie[2] thinks that the *Dioptra* is the work of a writer of the seventh or eighth century A.D., called *Heron the Younger*. But we have no reliable evidence that a second mathematician by the name of Heron really existed.[3] A reason adduced by Marie for the later origin of the *Dioptra* is the fact that it is the first work which contains the important formula for the area of a triangle, expressed in terms of the three sides. Now, not a single Greek writer cites this formula; hence he thinks it improbable that the *Dioptra* was written as early as the time of Heron the Elder. This argument is not convincing, for the reason that only a small part of Greek mathematical literature of this period has been preserved. The formula, sometimes called "Heronic formula," expresses the triangular area as follows:

$$\sqrt{\frac{a+b+c}{2}\cdot\frac{a+b-c}{2}\cdot\frac{a+c-b}{2}\cdot\frac{b+c-a}{2}}$$

where a, b, c are the sides. The proof given, though laborious,

[1] According to CANTOR, I., 363, he flourished about 100 B.C.; according to MARIE, I., 177, about 155 B.C.

[2] MARIE, I., 180.

[3] CANTOR, I., 367.

is exceedingly ingenious and discloses no little mathematical ability.[1]

"Dioptra," says Venturi, "were instruments resembling our modern theodolites. The instrument consisted of a rod four yards long with little plates at the ends for aiming. This rested upon a circular disc. The rod could be moved horizontally and also vertically. By turning the rod around until stopped by two suitably located pins on the circular disc, the surveyor could work off a line perpendicular to a given direction. The level and plumb-line were used."[2] Heron explains with aid of these instruments and of geometry a large number of problems, such as to find the distance between two points of which one only is accessible, or between two points which are visible, but both inaccessible; from a given point to run a perpendicular to a line which cannot be approached; to find the difference of level between two points; to measure the area of a field without entering it.

The *Dioptra* discloses considerable mathematical ability and familiarity with the writings of the authors of the classical period. Heron had read Hipparchus and he wrote a commentary on Euclid.[3] Nevertheless the character of his geometry is not Grecian, but Egyptian. Usually he gives directions and rules, without proofs. He gives formulæ for computing the area of a regular polygon from the square of one of its sides; this implies a knowledge of trigonometry. Some of Heron's formulæ point to an old Egyptian origin. Thus, besides the formula for a triangular area, given above, he gives $\frac{a_1+a_2}{2}\times\frac{b}{2}$, which bears a striking likeness to the formula $\frac{a_1+a_2}{2}\times\frac{b_1+b_2}{2}$,

[1] For the proof, see *Dioptra* (Ed. Hultsch), pp. 235–237; Cantor, I., 375; Gow, p. 281.

[2] Cantor, I., 382.

[3] See Tannery, pp. 165–181.

for finding the area of a quadrangle, found in the Edfu inscriptions. There are points of resemblance between Heron and Ahmes. Thus, Ahmes used unit-fractions exclusively; Heron used them oftener than other fractions. That the arithmetical theories of Ahmes were not forgotten at this time is also demonstrated by the Akhmîm papyrus, which, though the oldest extant text-book on practical Greek arithmetic, was probably written after Heron's time. Like Ahmes and the priests at Edfu, Heron divides complicated figures into simpler ones by drawing auxiliary lines; like them he displays particular fondness for the isosceles trapezoid.

The writings of Heron satisfied a practical want, and for that reason were widely read. We find traces of them in Rome, in the Occident during the Middle Ages, and even in India.

VI. *The Second Alexandrian School.* — With the absorption of Egypt into the Roman Empire and with the spread of Christianity, Alexandria became a great commercial as well as intellectual centre. Traders of all nations met in her crowded streets, while in her magnificent library, museums, and lecture-rooms, scholars from the East mingled with those of the West. Greek thinkers began to study the Oriental literature and philosophy. The resulting fusion of Greek and Oriental philosophy led to Neo-Pythagoreanism and Neo-Platonism. The study of Platonism and Pythagorean mysticism led to a revival of the theory of numbers. This subject became again a favourite study, though geometry still held an important place. This second Alexandrian school, beginning with the Christian era, was made famous by the names of Diophantus, Claudius Ptolemæus, Pappus, Theon of Smyrna, Theon of Alexandria, Iamblichus, Porphyrius, Serenus of Antinœia, Menelaus, and others.

Menelaus of Alexandria lived about 98 A.D., as appears from

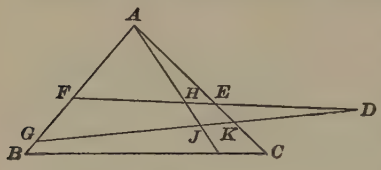
two astronomical observations taken by him and recorded in the *Almagest.*[1] Valuable contributions were made by him to spherical geometry, in his work, *Sphærica,* which is extant in Hebrew and Arabic, but lost in the original Greek. He gives the theorems on the congruence of spherical triangles and describes their properties in much the same way as Euclid treats plane triangles. He gives the theorems that the sum of the three sides of a spherical triangle is less than a great circle, and that the sum of three angles exceeds two right angles. Celebrated are two theorems of his on plane and spherical triangles. The one on plane triangles is that "if the three sides be cut by a straight line, the product of the one set of three segments which have no common extremity is equal to the product of the other three." The illustrious Lazare Carnot makes this proposition, known as the "lemma of Menelaus," the base of his theory of transversals.[2] The corresponding theorem for spherical triangles, the so-called "rule of six quantities," is obtained from the above by reading "chords of three segments doubled," in place of "three segments."

Another fundamental theorem in modern geometry (in the theory of harmonics) is the following ascribed to *Serenus* of Antinœia: If from D we draw DF, cutting the triangle ABC, and choose H on it, so that $DE : DF = HE : HF$, and if we draw the line AH, then every transversal through D, such as

[1] Cantor, I., 412.

[2] For the history of the theorem see M. Chasles, *Geschichte der Geometrie.* Aus dem Französischen übertragen durch Dr. L. A. Sohncke. Halle, 1839, Note VI., pp. 295–299. Hereafter we quote this work as Chasles. A recent French edition of this important work is now easily obtainable. Chasles points out that the "lemma of Menelaus" was well known during the sixteenth and seventeenth centuries, but from that time, for over a century, it was fruitless and hardly known, until finally Carnot began his researches. Carnot, as well as Ceva, rediscovered the theorem.

DG, will be divided by AH so that $DK : DG = JK : JG$. As Antinœia (or Antinoupolis) in Egypt was founded by Emperor Hadrian in 122 A.D., we have an upper limit for the date of Serenus.[1] The fact that he is quoted by a writer of the fifth or sixth century supplies us with a lower limit.



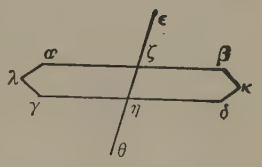
A central position in the history of ancient astronomy is occupied by *Claudius Ptolemæus*. Nothing is known of his personal history except that he was a native of Egypt and flourished in Alexandria in 139 A.D. The chief of his works are the *Syntaxis Mathematica* (or the *Almagest*, as the Arabs called it) and the *Geographica*. This is not the place to describe the "Ptolemaic System"; we mention Ptolemæus because of the geometry and especially the trigonometry contained in the *Almagest*. He divides the circle into 360 degrees; the diameter into 120 divisions, each of these into 60 parts, which are again divided into 60 smaller parts. In Latin, these parts were called *partes minutæ primæ* and *partes minutæ secundæ*.[2] Hence our names "minutes" and "seconds." Thus the Babylonian sexagesimal system, known to Geminus and Hipparchus, had by this time taken firm root among the Greeks in Egypt. The foundation of trigonometry had been laid by the illustrious Hipparchus. Ptolemæus imparted to it a remarkably perfect form. He calculated a table of chords by a method which seems original with him. After proving the proposition, now appended to Euclid, VI. (D), that "the rectangle contained by the diagonals of a quadrilateral figure inscribed in a circle is equal to both the rectangles contained by its opposite sides," he shows how to find from the chords of two arcs the chords of

[1] J. L. Heiberg in *Bibliotheca Mathematica*, 1894, p. 97.
[2] Cantor, I., p. 416.

their sum and difference, and from the chord of any arc that of its half. These theorems, to which he gives pretty proofs,[1] are applied to the calculation of chords. It goes without saying that the nomenclature and notation (so far as he had any) were entirely different from those of modern trigonometry. In place of our "sine," he considers the "chord of double the arc." Thus, in his table, the chord $21 \cdot 21 \cdot 12$ is given for the arc 20° 30′. Reducing the sexagesimals to decimals, we get for the chord .35588. Its half, .17794, is seen to be the *sine* of 10° 15′, or the half of 20° 30′.

More complete than in plane trigonometry is the theoretical exposition of propositions in spherical trigonometry.[2] Ptolemy starts out with the theorems of Menelaus. The fact that trigonometry was cultivated not for its own sake, but to aid astronomical inquiry, explains the rather startling fact that spherical trigonometry came to exist in a developed state earlier than plane trigonometry.

Of particular interest to us is a proof which, according to Proclus, was given by Ptolemy to Euclid's parallel-postulate. The critical part of the proof is as follows: If the straight lines are parallel, the interior angles [on the same side of the transversal] are necessarily equal to two right angles. For $\alpha\zeta$, $\gamma\eta$ are not less parallel than $\zeta\beta$, $\eta\delta$ and, therefore, whatever the sum of the angles $\beta\zeta\eta$, $\zeta\eta\delta$, whether greater or less than two right angles, such also must be the sum of the angles $\alpha\zeta\eta$, $\zeta\eta\gamma$. But the sum of the four cannot be more than four right angles, because they are two pairs of adjacent angles.[3] The untenable point in this proof



[1] Consult Cantor, I., 417, for the proofs.
[2] Consult Cantor, I., p. 420 ; Gow, p. 297.
[3] Gow, p. 301.

is the assertion that, in case of parallelism, the sum of the interior angles on one side of the transversal must be the same as their sum on the other side of the transversal. Ptolemy appears to be the first of a long line of geometers who during eighteen centuries vainly attempted to *prove* the parallel-postulate, until finally the genius of Lobatchewsky and Bolyai dispelled the haze which obstructed the vision of mathematicians and led them to see with unmistakable clearness the reason why such proofs have been and always will be futile.

For 150 years after Ptolemy, there appeared no geometer of note. An unimportant work, entitled *Cestes*, by *Sextus Julius Africanus*, applies geometry to the art of war. *Pappus* (about 300 or 370, A.D.) was the last great mathematician of the Alexandrian school. Though inferior in genius to Archimedes, Apollonius, and Euclid, who wrote five centuries earlier, Pappus towers above his contemporaries as does a lofty peak above the surrounding plains. Of his several works the *Mathematical Collections* is the only one extant. It was in eight books; the first and parts of the second are now missing. The object of this treatise appears to have been to supply geometers with a succinct analysis of the most difficult mathematical works and to facilitate the study of these by explanatory lemmas. It is invaluable to us on account of the rich information it gives on various treatises by the foremost Greek mathematicians, which are now lost. Scholars of the last century considered it possible to restore lost works from the *résumé* given by Pappus alone. Some of the theorems are doubtless original with Pappus, but here it is difficult to speak with certainty, for in three instances Pappus copied theorems without crediting them to the authors, and he may have done the same in other cases where we have no means of ascertaining the real discoverer.

Of elementary propositions of special interest and probably

his own, we mention the following: (1) The centre of inertia (gravity) of a triangle is that of another triangle whose vertices lie upon the sides of the first and divide its three sides in the same ratio; (2) solution of the problem to draw through three points lying in the same straight line, three straight lines which shall form a triangle inscribed in a given circle;[1] (3) he propounded the theory of involution of points; (4) the line connecting the opposite extremities of parallel diameters of two externally tangent circles passes through the point of contact (a theorem suggesting the consideration of centres of similitude of two circles); (5) solution of the problem, to find a parallelogram whose sides are in a fixed ratio to those of a given parallelogram, while the areas of the two are in another fixed ratio. This resembles somewhat an indeterminate problem given by Heron, to construct two rectangles in which the sums of their sides, as well as their areas, are in given ratios.[2]

It remains for us to name a few more mathematicians. *Theon of Alexandria* brought out an edition of Euclid's *Elements*, with notes; his commentary on the *Almagest* is valuable on account of the historical notes and the specimens of Greek arithmetic found therein. Theon's daughter *Hypatia*, a woman renowned for her beauty and modesty, was the last Alexandrian teacher of reputation. Her notes on Diophantus and Apollonius have been lost. Her tragic death in 415 A.D. is vividly described in Kingsley's *Hypatia*.[3]

From now on Christian theology absorbed men's thoughts.

[1] "The problem, generalized by placing the points anywhere, has become celebrated, partly by its difficulty, partly by the names of geometers who solved it, and especially by the solution, as general as it was simple, given by a boy of 16, Ottaiano of Naples." CHASLES, p. 41; Chasles gives a history of the problem in *note* XI. [2] CANTOR, I., 454.

[3] The reader may be interested in an article by G. VALENTIN, "Die Frauen in den exakten Wissenschaften," *Bibliotheca Mathematica*, 1895, pp. 65–76.

In Alexandria Paganism disappeared and with it Pagan learning. The Neo-Platonic school at Athens struggled a century longer. Proclus, Isidorus, and others endeavoured to keep up "the golden chain of Platonic succession." *Proclus* wrote a commentary on Euclid; that on the first book is extant and is of great historical value. A pupil of Isidorus, *Damascius* of Damascus (about 510 A.D.) is believed by some to be the author of Book XV. of Euclid's *Elements*.

The geometers of the last 500 years, with the possible exception of Pappus, lack creative power; they are commentators rather than originators.

The salient features of Greek geometry are:

(1) A wonderful clearness and definiteness of concepts, and an exceptional logical rigour of conclusions. We have encountered occasional flaws in reasoning; but when we compare Greek geometry in its most complete form with the best that the Babylonians, Egyptians, Romans, Hindus, or the geometers of the Middle Ages have brought forth, then it must be admitted that not only in rigour of presentation, but also in fertility of invention, the geometric mind of the Greek towers above all others in solitary grandeur.

(2) A complete absence of general principles and methods. For example, the Greeks possessed no general method of drawing tangents. In the demonstration of a theorem, there were, for the ancient geometers, as many different cases requiring separate proof as there were different positions for the lines.[1] "It is one of the greatest advantages of the more modern geometry over the ancient that through the consideration of positive and negative quantities it embraces in *one* expression the several cases which a theorem can present in respect to the various relative positions of the separate parts

[1] Take, for example, Euclid, III., 35.

of the figure. Thus to-day the nine main problems and the numerous special cases, which are the subject-matter of 83 theorems in the two books *de sectione determinata* (of Pappus), constitute only *one* problem which can be solved by *one single* equation."[1] "If we compare a mathematical problem with a huge rock, into the interior of which we desire to penetrate, then the work of the Greek mathematicians appears to us like that of a vigorous stonecutter who, with chisel and hammer, begins with indefatigable perseverance, from without, to crumble the rock slowly into fragments; the modern mathematician appears like an excellent miner, who first bores through the rock some few passages, from which he then bursts it into pieces with one powerful blast, and brings to light the treasures within."[2]

ROME

Although the Romans excelled in the science of government and war, in philosophy, poetry, and art they were mere imitators. In mathematics they did not even rise to the desire for imitation. If we except the period of decadence, during which the reading of Euclid began, we can say that the classical Greek writers on geometry were wholly unknown in Rome. A science of geometry with definitions, postulates, axioms, rigorous proofs, did not exist there. A *practical* geometry, like the old Egyptian, with empirical rules applicable in surveying, stood in place of the Greek science. Practical treatises prepared by Roman surveyors, called *agrimensores* or *gromatici*, have come down to us. "As regards the geometrical part of

[1] Chasles, p. 39.

[2] Hermann Hankel, *Die Entwickelung der Mathematik in den letzten Jahrhunderten*, Tübingen, 1884, p. 9.

these pandects, which treat exhaustively also of the juristic and purely technical side of the art, it is difficult to say whether the crudeness of presentation, or the paucity and faultiness of the contents more strongly repels the reader. The presentation is beneath the notice of criticism, the terminology vacillating; of definitions and axioms or proofs of the prescribed rules there is no mention. The rules are not formulated; the reader is left to abstract them from numerical examples obscurely and inaccurately described. The total impression is as though the Roman gromatici were thousands of years older than Greek geometry, and as though the deluge were lying between the two."[1] Some of their rules were probably inherited from the Etruscans, but others are identical with those of Heron. Among the latter is that for finding the area of a triangle from its sides [the "Heronic formula"] and the approximate formula, $\frac{13}{80}a^2$, for the area of equilateral triangles (a being one of the sides). But the latter area was also calculated by the formulas $\frac{1}{2}(a^2+a)$ and $\frac{1}{2}a^2$, the first of which was unknown to Heron. Probably $\frac{1}{2}a^2$ was derived from an Egyptian formula. The more elegant and refined methods of Heron were unknown to the Romans. The gromatici considered it sometimes sufficiently accurate to determine the areas of cities irregularly laid out, simply by measuring their circumferences.[2] Egyptian geometry, or as much of it as the Romans thought they could use, was imported at the time of Julius Cæsar, who ordered a survey of the whole empire to secure an equitable mode of taxation. From early times it was the Roman practice to divide land into rectangular and rectilinear parts. Walls and streets were parallel, enclosing squares of prescribed dimensions. This practice

[1] Hankel, pp. 295, 296. For a detailed account of the *agrimensores*, consult Cantor, I., pp. 521–538.

[2] Hankel, p. 297.

simplified matters immensely and greatly reduced the necessary amount of geometrical knowledge. Approximate formulæ answered all ordinary demands of precision.

Cæsar reformed the calendar, and for this undertaking drew from Egyptian learning. The Alexandrian astronomer *Sosigenes* was enlisted for this task. Among Roman names identified with geometry or surveying, are the following: *Marcus Terentius Varro* (about 116–27 B.C.), *Sextus Julius Frontinus* (in 70 A.D. prætor in Rome), *Martianus Mineus Felix Capella* (born at Carthage in the early part of the fifth century), *Magnus Aurelius Cassiodorius* (born about 475 A.D.). Vastly superior to any of these were the Greek geometers belonging to the period of decadence of Greek learning.

It is a remarkable fact that the period of political humiliation, marked by the fall of the Western Roman Empire and the ascendancy of the Ostrogoths, is the period during which the study of Greek science began in Italy. The compilations made at this time are deficient, yet interesting from the fact that, down to the twelfth century, they were the only sources of mathematical knowledge in the Occident. Foremost among these writers is *Anicius Manlius Severinus Boethius* (480?–524). At first a favourite of King Theodoric, he was later charged with treason, imprisoned, and finally decapitated. While in prison he wrote *On the Consolations of Philosophy.* Boethius wrote an *Institutio Arithmetica* (essentially a translation of the arithmetic of Nicomachus) and a *Geometry.* The first book of his *Geometry* is an extract from the first three books of Euclid's *Elements,* with the proofs omitted. It appears that Boethius and a number of other writers after him were somehow led to the belief that the theorems alone belonged to Euclid, while the proofs were interpolated by Theon; hence the strange omission of all demonstration. The second book in the *Geometry* of Boethius

consists of an abstract of the practical geometry of Frontinus, the most accomplished of the gromatici.

Notice that, imitating Nicomachus, Boethius divides the mathematical sciences into four sections, Arithmetic, Music, Geometry, Astronomy. He first designated them by the word *quadruvium* (four path-ways). This term was used extensively during the Middle Ages. Cassiodorius used a similar figure, the four gates of science. Isidorus of Carthage (born 570), in his *Origines*, groups all sciences as seven, the four embraced by the *quadruvium* and three (Grammar, Rhetoric, Logic) which constitute the *trivium* (three path-ways).

MIDDLE AGES

ARITHMETIC AND ALGEBRA

Hindus

Soon after the decadence of Greek mathematical research, another Aryan race, the Hindus, began to display brilliant mathematical power. Not in the field of geometry, but of arithmetic and algebra, they achieved glory. In geometry they were even weaker than were the Greeks in algebra. The subject of indeterminate analysis (not within the scope of this history) was conspicuously advanced by them, but on this point they exerted no influence on European investigators, for the reason that their researches did not become known in the Occident until the nineteenth century.

India had no professed mathematicians; the writers we are about to discuss considered themselves astronomers. To them, mathematics was merely a handmaiden to astronomy. In view of this it is curious to observe that the auxiliary science is after all the only one in which they won real distinction, while in their pet pursuit of astronomy they displayed an inaptitude to observe, to collect facts, and to make inductive investigations.

It is an unpleasant feature about the Hindu mathematical treatises handed down to us that rules and results are expressed in verse and clothed in obscure mystic language. To him who

already understands the subject such verses may aid the memory, but to the uninitiated they are often unintelligible. Usually proofs are not preserved, though Hindu mathematicians doubtless reasoned out all or most of their discoveries.

It is certain that portions of Hindu mathematics are of Greek origin. An interesting but difficult task is the tracing of the relation between Hindu and Greek thought. After Egypt had become a Roman province, extensive commercial relations sprang up with Alexandria. Doubtless there was considerable interchange of philosophic and scientific knowledge. The Hindus were indebted to Heron, Diophantus and Ptolemy. They were borrowers also from the Chinese.

At present we know very little of the growth of Hindu mathematics. The few works handed down to us exhibit the science in its complete state only. The dates of all important works but the first are well fixed. In 1881 there was found at Bakhshâlî, in northwest India, buried in the earth, an anonymous arithmetic, supposed, from the peculiarities of its verses, to date from the third or fourth century after Christ. The document that was found is of birch bark and is an incomplete copy, prepared probably about the eighth century, of an older manuscript.[1]

The earliest Hindu astronomer known to us is *Aryabhatta*, born in 476 A.D., at Pataliputra, on the upper Ganges. He is the author of the celebrated work, entitled *Aryabhattiyam*, the third chapter of which is given to mathematics.[2] About one hundred years later flourished *Brahmagupta*, born 598, who in 628 wrote his *Brahma-sphuta-siddhanta* ("The revised system of Brahma"), of which the twelfth and eighteenth chapters

[1] CANTOR, I., pp. 598, 613–615. See also *The Bakhshâlî Manuscript*, edited by RUDOLF HOERNLE in the *Indian Antiquary*, XVII., 33–48 and 275–279, Bombay, 1888.

[2] Translated by L. RODET in *Journal Asiatique*, 1879, série 7, T. XIII.

belong to mathematics.[1] Important is an elementary work by Mahavira (9th century ?); then come, *Çridhara*, who wrote a *Ganita-sara* ("Quintessence of Calculation"), and *Padmanabha*, the author of an algebra. The science seems to have made but little progress since Brahmagupta, for a work entitled *Siddhantaciromani* ("Diadem of an Astronomical System"), written by *Bhaskara Acarya* in 1150, stands little higher than Brahmagupta's *Siddhanta*, written over five hundred years earlier. The two most important mathematical chapters in Bhaskara's work are the *Lilavati* ("the beautiful," *i.e.* the noble science), and *Viya-ganita* ("root-extraction"), devoted to arithmetic and algebra. From this time the spirit of research was extinguished and no great names appear.

Elsewhere we have spoken of the Hindu's clever use of the principle of local value and of the zero in arithmetical notation. We now give an account of Hindu methods of computation, which were elaborated in India to a perfection undreamed of by earlier nations. The information handed down to our time is derived partly from the Hindu works, but mainly from an arithmetic written by a Greek monk, *Maximus Planudes*, who lived in the first half of the fourteenth century, and who avowedly used Hindu sources.

To understand the reason why certain modes of computation were adopted, we must bear in mind the instruments at the disposal of the Hindus in their written calculations. They

[1] Translated into English by H. T. Colebrooke, London, 1817. This celebrated Sanskrit scholar translated also the mathematical chapters of the *Siddhantaciromani* of Bhaskara. Colebrooke held a judicial office in India, and about the year 1794 entered upon the study of Sanskrit, that he might read Hindu law-books. From youth fond of mathematics, he also began to study Hindu astronomy and mathematics; and, finally, by his translations, demonstrated to Europe the fact that the Hindus in previous centuries had made remarkable discoveries, some of which had been wrongly ascribed to the Arabs.

wrote "with a cane pen upon a small blackboard with a white, thinly-liquid paint which made marks that could be easily erased, or upon a white tablet, less than a foot square, strewn with red flour, on which they wrote the figures with a small stick, so that the figures appeared white on a red ground."[1] To be legible the figures had to be quite large, hence it became necessary to devise schemes for the saving of space. This was accomplished by erasing a digit as soon as it had done its service. The Hindus were generally inclined to follow the motion from left to right, as in writing. Thus in adding 254 and 663, they would say $2 + 6 = 8$, $5 + 6 = 11$, which changes 8 to 9, $4 + 3 = 7$. Hence the sum 917.

In *subtraction*, they had two methods when "borrowing" became necessary. Thus, in $51 - 28$, they would say 8 from $11 = 3$, 2 from $4 = 2$; or they would say 8 from $11 = 3$, 3 from $5 = 2$.

In *multiplication* several methods were in vogue. Sometimes they resolved the multiplier into its factors, and then multiplied in succession by each factor. At other times they would resolve the multiplier into the sum or the difference of two numbers which were easier multipliers. In written work, a multiplication, such as 5×57893411, was done thus: $5 \times 5 = 25$, which was written down above the multiplicand; $5 \times 7 = 35$; adding 3 to 25 gives 28; erase the 5 and in its place write 8. We thus have 285. Then, $5 \times 8 = 40$; $4 + 5 = 9$; replace 5 by 9, and we have 2890, etc. At the close of the operation the work on the tablet appeared somewhat as follows:

289467055
57893411 5

When the multiplier consisted of several digits, then the Hindu operation, as described by Hankel (p. 188), in case of

[1] HANKEL, p. 186.

324×753, was as follows: Place the left-hand digit of the multiplier, 324, over units' place in the multiplicand; $3 \times 7 = 21$, write this down; $3 \times 5 = 15$; replace 21 by 22; $3 \times 3 = 9$. At this stage the appearance of the work is as here indicated. Next, the multiplicand is moved one place to the right. $2 \times 7 = 14$; in the place where the 14 belongs we already have 25, the two together give 39, which is written in place of 25; $2 \times 5 = 10$; add 10 to 399, and write 409 in place of 399; $2 \times 3 = 6$. The adjoining figure shows the work at this stage. We begin the third step by again moving the multiplicand to the right one place; $4 \times 7 = 28$; add this to 09 and write down 37 in its place, etc.

2259
324
753

24096
324
753

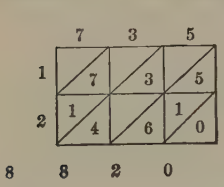
243972
324
753

This method, employed by Hindus even at the present time, economizes space remarkably well, since only a few of all the digits used appear on the tablet at any one moment. It was, therefore, well adapted to their small tablets and coarse pencils. The method is a poor one, if the calculation is to be done on paper, (1) because we cannot readily and neatly erase the digits, and (2) because, having plenty of paper, it is folly to save space and thereby unnecessarily complicate the process by performing the addition of each partial product, as soon as formed. Nevertheless, we find that the early Arabic writers, unable to improve on the Hindu process, adopted it and showed how it can be carried out on paper, viz., by crossing out the digits (instead of erasing them) and placing the new digits above the old ones.[1]

Besides these, the Hindus had other methods, more closely resembling the processes in vogue at the present time. Thus a tablet was divided into squares like a chess-board. Diagonals were drawn. The multiplication of $12 \times 735 = 8820$ is ex-

[1] HANKEL, p. 188.

hibited in the adjoining diagram.[1] The manuscripts extant give no detailed information regarding the Hindu process of *division*. It seems that the partial products were deducted by erasing digits in the dividend and replacing them by the new ones resulting from the subtraction. The above scheme of multiplication was known to the Arabs and may not be of Hindu origin.



The Hindus used an ingenious process (not of Indian origin) for testing the correctness of their computations. It rests on the theorem that the sum of the digits of a number, divided by 9, gives the same remainder as does the number itself, divided by 9. The process of "casting out the 9's" was more serviceable to the Hindus than it is to us. Their custom of erasing digits and writing others in their places made it much more difficult for them to verify their results by reviewing the operations performed. At the close of a multiplication a large number of the digits arising during the process were erased. Hence a test which did not call for the examination of the intermediate processes was of service to them.

In the extant fragments of the *Bakhshâlî arithmetic*, a knowledge of the processes of computation is presupposed. In fractions, the numerator is written above the denominator without a dividing line. Integers are written as fractions with the denominator 1. In mixed expressions the integral part is written above the fraction. Thus, $\overset{1}{1} = 1\frac{1}{3}$. In place of our = they used the word *phalam*, abbreviated into *pha*. Addition was indicated by *yu*, abbreviated from *yuta*. Numbers to be combined were often enclosed in a rectangle. Thus, *pha* 12 $\boxed{\begin{matrix}5 & 7\\ 1 & 1\end{matrix}}$ *yu* means $\frac{5}{1} + \frac{7}{1} = 12$. An unknown

[1] CANTOR, I., p. 611.

quantity is *sunya*, and is designated thus • by a heavy dot. The word sunya means "empty," and is the word for zero, which is here likewise represented by a dot. This double use of the word and dot rested upon the idea that a position is "empty" if not filled out. It is also to be considered "empty" so long as the number to be placed there has not been ascertained.[1]

The Bakhshâlî arithmetic contains problems of which some are solved by reduction to unity or by a sort of *false position*. EXAMPLE: B gives twice as much as A, C three times as much as B, D four times as much as C; together they give 132; how much did A give? Take 1 for the unknown (*sunya*), then $A = 1$, $B = 2$, $C = 6$, $D = 24$, their sum $= 33$. Divide 132 by 33, and the quotient 4 is what A gave.

The method of *false position* we have encountered before among the early Egyptians. With them it was an instinctive procedure; with the Hindus it had risen to a conscious method. Bhaskara uses it, but while the Bakhshâlî document preferably assumes 1 as the unknown, Bhaskara is partial to 3. Thus, if a certain number is taken five-fold, $\frac{1}{3}$ of the product be subtracted, the remainder divided by 10, and $\frac{1}{3}$, $\frac{1}{2}$, and $\frac{1}{4}$ of the original number added, then 68 is obtained. What is the number? Choose 3, then you get 15, 10, 1, and $1 + \frac{3}{3} + \frac{3}{2} + \frac{3}{4} = \frac{17}{4}$. Then $(68 \div \frac{17}{4})\, 3 = 48$, the answer.[2]

A favourite method of solution is that of *inversion*. With laconic brevity, Aryabhatta describes it thus: "Multiplication becomes division, division becomes multiplication; what was gain becomes loss, what loss, gain; inversion." Quite different from this in style is the following problem of Aryabhatta, which illustrates the method: "Beautiful maiden with beaming eyes, tell me, as thou understandst the right

[1] CANTOR, I., pp. 613–615.

[2] CANTOR, I., p. 618.

method of inversion, which is the number which multiplied by 3, then increased by $\frac{3}{4}$ of the product, divided by 7, diminished by $\frac{1}{3}$ of the quotient, multiplied by itself, diminished by 52, by extraction of the square root, addition of 8, and division by 10, gives the number 2?" The process consists in beginning with 2 and working backwards. Thus, $(2 \cdot 10 - 8)^2 + 52 = 196$, $\sqrt{196} = 14$, and $14 \cdot \frac{3}{2} \cdot 7 \cdot \frac{4}{7} \div 3 = 28$, the answer.[1] Here is another example taken from the *Lilavati:* "The square root of half the number of bees in a swarm has flown out upon a jessamine-bush, $\frac{8}{9}$ of the whole swarm has remained behind; one female bee flies about a male that is buzzing within a lotus-flower into which he was allured in the night by its sweet odour, but is now imprisoned in it. Tell me the number of bees."[2] Answer 72. The pleasing poetic garb in which arithmetical problems are clothed is due to the Hindu practice of writing all school-books in verse, and especially to the fact that these problems, propounded as puzzles, were a favourite social amusement. Says Brahmagupta: "These problems are proposed simply for pleasure; the wise man can invent a thousand others, or he can solve the problems of others by the rules given here. As the sun eclipses the stars by his brilliancy, so the man of knowledge will eclipse the fame of others in assemblies of the people if he proposes algebraic problems, and still more if he solves them."

The Hindus were familiar with the Rule of Three, with the computation of interest (simple and compound), with alligation, with the fountain or pipe problems, and with the summation of arithmetic and geometric series. Aryabhatta applies the Rule of Three to the problem — a 16 year old girl slave costs 32 nishkas, what costs one 20 years old? — and says that it is treated by inverse proportion, since "the value

[1] Cantor I., p. 618. [2] Hankel, p. 191.

of living creatures (slaves and cattle) is regulated by their age," the older being the cheaper.[1] The extraction of square and cube roots was familiar to the Hindus. This was done with aid of the formulas

$$(a+b)^2 = a^2 + 2ab + b^2;\ (a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

The Hindus made remarkable contributions to Algebra. Addition was indicated simply by juxtaposition as in Diophantine algebra; subtraction, by placing a dot over the subtrahend; multiplication, by putting after the factors, *bha*, the abbreviation of the word *bhavita*, "the product"; division, by placing the divisor beneath the dividend; square root, by writing *ka*, from the word *karana* (irrational), before the quantity. The unknown quantity was called by Brahmagupta *yâvattâvat*. In case of several unknown quantities, he gave, unlike Diophantus, a distinct name and symbol to each. Unknown quantities after the first were assigned the names of colours, being called the black, blue, yellow, red, or green unknown. The initial syllable of each word was selected as the symbol for the unknown. Thus *yâ* meant x; *kâ* (from *kâlaka* = black) meant y; *yâ kâ bha*, "x times y"; *ka* 15 *ka* 10, "$\sqrt{15} - \sqrt{10}$."

The Hindus were the first to recognize the existence of absolutely negative numbers and of irrational numbers. The difference between + and − numbers was brought out by attaching to the one the idea of "assets," to the other that of "debts," or by letting them indicate opposite directions. A great step beyond Diophantus was the recognition of two answers for quadratic equations. Thus Bhaskara gives $x = 50$ or -5 for the roots of $x^2 - 45x = 250$. "But," says he, "the second value is in this case not to be taken, for it is inadequate;

[1] Cantor I., p. 618.

people do not approve of negative roots." Thus negative roots were seen, but not admitted. In the Hindu mode of solving quadratic equations, as found in Brahmagupta and Aryabhatta, it is believed that Greek processes are discernible. For example, in their works, as also in Heron of Alexandria, $ax^2 + bx = c$ is solved by a rule yielding

$$x = \frac{\sqrt{ac + \left(\frac{b}{2}\right)^2} - \frac{b}{2}}{a}.$$

This rule is improved by Çridhara, who begins by multiplying the members of the equation, not by a as did his predecessors, but by $4a$, whereby the possibility of fractions under the radical sign is excluded. He gets

$$x = \frac{\sqrt{4ac + b^2} - b}{2a}.$$

The most important advance in the theory of affected quadratic equations made in India is the unifying under one rule of the three cases

$$ax^2 + bx = c, \quad bx + c = ax^2, \quad ax^2 + c = bx.$$

In Diophantus these cases seem to have been dealt with separately, because he was less accustomed to deal with negative quantities.[1]

An advance far beyond the Greeks and even beyond Brahmagupta is the statement of Bhaskara that "the square of a positive, as also of a negative number, is positive; that the square root of a positive number is twofold, positive and negative. There is no square root of a negative number, for it is not a square."

We have seen that the Greeks sharply discriminated between numbers and magnitudes, that the irrational was not recognized

[1] Cantor, I., p. 625.

by them as a number. The discovery of the existence of irrationals was one of their profoundest achievements. To the Hindus this distinction between the rational and irrational did not occur; at any rate, it was not heeded. They passed from one to the other, unmindful of the deep gulf separating the continuous from the discontinuous. Irrationals were subjected to the same processes as ordinary numbers and were indeed regarded by them as numbers. By so doing they greatly aided the *progress* of mathematics; for they accepted results arrived at intuitively, which by severely logical processes would call for much greater effort. Says Hankel (p. 195), "if one understands by algebra the application of arithmetical operations to complex magnitudes of all sorts, whether rational or irrational numbers or space-magnitudes, then the learned Brahmins of Hindostan are the real inventors of algebra."

In Bhaskara we find two remarkable identities, one of which is given in nearly all our school algebras, as showing how to find the square root of a "binomial surd." What Euclid in Book X. embodied in abstract language, difficult of comprehension, is here expressed to the eye in algebraic form and applied to numbers:[1]

$$\sqrt{a \pm \sqrt{b}} = \sqrt{\frac{a + \sqrt{a^2 - b}}{2}} \pm \sqrt{\frac{a - \sqrt{a^2 - b}}{2}}$$

$$\sqrt{a} \pm \sqrt{b} = \sqrt{a + b \pm 2\sqrt{ab}}.$$

Arabs

The Arabs present an extraordinary spectacle in the history of civilization. Unknown, ignorant, and disunited tribes of the Arabian peninsula, untrained in government and war, are

[1] HANKEL, p. 194.

in the course of ten years fused by the furnace blast of religious enthusiasm into a powerful nation, which, in one century, extended its dominions from India across northern Africa to Spain. A hundred years after this grand march of conquest, we see them assume the leadership of intellectual pursuits; the Moslems became the great scholars of their time.

About 150 years after Mohammed's flight from Mecca to Medina, the study of Hindu science was taken up at Bagdad in the court of Caliph Almansur. In 773 A.D. there appeared at his court a Hindu astronomer with astronomical tables which by royal order were translated into Arabic. These tables, known by the Arabs as the *Sindhind*, and probably taken from the *Siddhânta* of Brahmagupta, stood in great authority. With these tables probably came the Hindu numerals. Except for the travels of Albîrûnî, we possess no other evidence of intercourse between Hindu and Arabic scholars; yet we should not be surprised if future historical research should reveal greater intimacy. Better informed are we as to the Arabic acquisition of Greek learning. Elsewhere we shall speak of geometry and trigonometry. *Abû'l Wafâ* (940–998) translated the treatise on algebra by Diophantus, one of the last of the Greek authors to be brought out in Arabic. Euclid, Apollonius, and Ptolemæus had been acquired by the Arabic scholars at Bagdad nearly two centuries earlier.

Of all Arabic arithmetics known to us, first in time and first in historical importance is that of *Muḥammed ibn Mûsâ Alchwarizmî* who lived during the reign of Caliph Al Mamûn (813–833). Like all Arabic mathematicians whom we shall name, he was first of all an astronomer; with the Arabs as with the Hindus, mathematical pursuits were secondary. Alchwarizmî's arithmetic was supposed to be lost, but in 1857 a Latin translation of it, made probably by Athelard of Bath,

was found in the library of the University of Cambridge.[1] The arithmetic begins with the words, "Spoken has Algoritmi. Let us give deserved praise to God, our leader and defender." Here the name of the author, *Alchwarizmî*, has passed into *Algoritmi*, whence comes our modern word *algorithm*, signifying the art of computing in any particular way. Alchwarizmî is familiar with the principle of local value and Hindu processes of calculation. According to an Arabic writer, his arithmetic "excels all others in brevity and easiness, and exhibits the Hindu intellect and sagacity in the grandest inventions."[2] Both in *addition* and *subtraction* Alchwarizmî proceeds from left to right, but in subtraction, strange to say, he fails to explain the case when a larger digit is to be taken from a smaller. His *multiplication* is one of the Hindu processes, modified for working on paper: each partial product is written *over* the corresponding digit in the multiplicand; digits are not erased as with the Hindus, but are crossed out. The process of *division* rests on the same idea. The divisor is written below the dividend, the quotient above it. The changes in the dividend resulting from the subtraction of the partial products are written above the quotient. For every new step in the division, the divisor is moved to the right one place. The author gives a lengthy description of the process in case of $46468 \div 324 = 143\frac{136}{324}$, which is represented by Cantor[3] by the adjoining model solution. This process of division was used almost exclusively by early European writers who followed Arabic models, and it

136	
24	
110	
22	
140	
143	
46468	
324	
324	
324	

[1] It was found by Prince B. Boncompagni under the title "Algoritmi de numero Indorum." He published it in his book *Trattati d' Aritmetica*, Rome, 1857.

[2] Cantor, I., 713; Hankel, p. 256.

[3] Cantor, I., 717. This process of division will be explained more fully under Pacioli.

was not extinct in Europe in the eighteenth century.[1] Later Arabic authors modified Alchwarizmî's process and thereby approached nearer to those now prevalent. Alchwarizmî explains in detail the use of sexagesimal fractions.

Arabic arithmetics usually explained, besides the four cardinal operations, the process of "casting out the 9's" (called sometimes the "Hindu proof"), the rule of "false position," the rule of "double position," square and cube root, and fractions (written without the fractional line, as by the Hindus). The Rule of Three occurs in Arabic works, sometimes in their algebras. It is a remarkable fact that among the early Arabs no trace whatever of the use of the abacus can be discovered. At the close of the thirteenth century, for the first time, we find an Arabic writer, *Ibn Albannâ*, who uses processes which are a mixture of abacal and Hindu computation. Ibn Albannâ lived in Bugia, an African seaport, and it is plain that he came under European influences and thence got a knowledge of the abacus.[2]

It is noticeable that in course of time, both in arithmetic and algebra, the Arabs in the East departed further and further from Hindu teachings and came more thoroughly under the influence of Greek science. This is to be deplored, for on these subjects the Hindus had new ideas, and by rejecting them the Arabs barred against themselves the road of progress. Thus, *Al Karchî* of Bagdad, who lived in the beginning of the eleventh century, wrote an arithmetic in which Hindu numerals are excluded! All numbers in the text are written out fully in words. In other respects the work is modelled almost entirely after Greek patterns. Another prominent and

[1] Hankel, p. 258.

[2] Whether the Arabs knew the use of the abacus or not is discussed also by H. Weissenborn, *Einführung der jetzigen Ziffern in Europa durch Gerbert*, pp. 5–9.

able writer, *Abû'l Wafâ*, in the second half of the tenth century, wrote an arithmetic in which Hindu numerals find no place. The question why Hindu numerals are ignored by authors so eminent, is certainly a puzzle. Cantor suggests that at one time there may have been rival schools, of which one followed almost exclusively Greek mathematics, the other Indian.[1]

The algebra of Alchwarizmî is the first work in which the word "algebra" occurs. The title of the treatise is *aldschebr walmukâbala*. These two words mean "restoration and opposition." By "restoration" was meant the transposing of negative terms to the other side of the equation; by "opposition," the discarding from both sides of the equation of like terms so that, after this operation, such terms appear only on that side of the equation on which they were in excess. Thus, $5x^2 - 2x = 6 + 3x^2$ passes by aldschebr into $5x^2 = 6 + 2x + 3x^2$; and this, by walmukâbala into $2x^2 = 6 + 2x$. When Alchwarizmî's *aldschebr walmukâbala* was translated into Latin, the Arabic title was retained, but the second word was gradually discarded, the first word remaining in the form of *algebra*. Such is the origin of this word, as revealed by the study of manuscripts. Several popular etymologies of the word, unsupported by manuscript evidence, used to be current. For instance, "algebra" was at one time derived from the name of the Arabic scholar, *Dschâbir ibn Aflah*, of Seville, who was called *Geber* by the Latins. But Geber lived two centuries after Alchwarizmî, and therefore two centuries after the first appearance of the word.[2]

Alchwarizmî's algebra, like his arithmetic, contains nothing

[1] Cantor, I., 765.

[2] See Cantor, I., 719, 721, 722; Hankel, p. 248; Felix Müller, *Historisch-etymologische Studien über Mathematische Terminologie*, Berlin, 1887, pp. 9, 10.

original. It explains the elementary operations and the solution of linear and quadratic equations. Whence did the author receive his knowledge of algebra? That he got it solely from Hindu sources is impossible, for the Hindus had no such rules as those of "restoration" and "opposition"; they were not in the habit of making all terms in an equation positive as is done by "restoration." Alchwarizmî's rules resemble somewhat those of Diophantus. But we cannot conclude that our Arabic author drew entirely from Greek sources, for, unlike Diophantus, but like the Hindus, he recognized two roots to a quadratic and accepted irrational solutions. It would seem, therefore, that the *aldschebr walmukâbala* was neither purely Greek nor purely Hindu, but was a hybrid of the two, with the Greek element predominating.

In one respect this and other Arabic algebras are inferior to both the Hindu and the Diophantine models: the Eastern Arabs use no symbols whatever. With respect to notation, algebras have been divided into three classes:[1] (1) *Rhetorical Algebras,* in which no symbols are used, everything being written out in words. Under this head belong Arabic works (excepting those of the later Western Arabs), the Greek works of Iamblichus and Thymaridas, and the works of the early Italian writers and of Regiomontanus. The equation $x^2 + 10x = 39$ was indicated by Alchwarizmî as follows: "A square and ten of its roots are equal to thirty-nine dirhem; that is, if you add to a square ten roots, then this together equals thirty-nine."

(2) *Syncopated Algebras,* in which, as in the first class, everything is written out in words, except that the abbreviations are used for certain frequently recurring operations and ideas. Such are the works of Diophantus, those of the later Western

[1] Nesselmann, pp. 302–306.

Arabs, and of the later European writers down to about the middle of the seventeenth century (excepting Vieta's). As an illustration we take a sentence from Diophantus, in which, for the sake of clearness, we shall use the Hindu numerals and, in place of the Greek symbols, the corresponding English abbreviations. If S., N., U., m. stand for "square," "number," "unity," "minus," then the solution of problem III., 7, in Diophantus, viz., to find three numbers whose sum is a square and such that any pair is a square, is as follows: "Let us assume the sum of the three numbers to be equal to the square[1] 1 S. 2 N. 1 U., the first and the second together 1 S. then the remainder 2 N. 1 U. will be the third number. Let the second and the third be equal to 1 S. 1 U. m. 2 N., of which the root is 1 N. m. 1 U. Now all three numbers are 1 S. 2 N. 1 U.; hence the first will be 4 N. But this and the second together were put equal to 1 S., hence the second will be 1 S. minus 4 N. Consequently, the first and third together, 6 N. 1 U. must be a square. Let this number be 121 U., then the number becomes 20 U. Hence the first is 80 U., the second 320 U., the third 41 U., and they satisfy the conditions."

(3) *Symbolic Algebras,* in which all forms and operations are represented by a fully developed symbolism, as, for example, $x^2 + 10x = 39$. In this class may be reckoned Hindu works as well as European since the middle of the seventeenth century.

From this classification due to Nesselmann the advanced ground taken by the Hindus is brought to full view; also the step backward taken by the early Arabs. The Arabs, however, made substantial contributions to what we may call geometrical algebra. Not only did they (Alchwarizmî, Al

[1] In our notation the expressions given here are respectively, x^2+2x+1, x^2, $2x+1$, x^2+1-2x, $x-1$, x^2+2x+1, $4x$, x^2, x^2-4x, $6x+1$.

Karchi) give geometrical proofs, besides the arithmetical, for the solution of quadratic equations, but they (Al Mâhânî, Abû Dscha 'far Alchâzin, Abû'l Dschûd, 'Omar Alchaijâmî) discovered a geometrical solution of cubic equations, which algebraically were still considered insolvable. The roots were constructed by intersecting conics.[1]

Al Karchî was the first Arabic author to give and prove the theorems on the summation of the series:

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{2n+1}{3}(1 + 2 + \cdots + n)$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + \cdots + n)^2$$

Allusion has been made to the fact that the Western Arabs developed an algebraic symbolism. While in the East the geometric treatment of algebra had become the fashion, in the West the Arabs elaborated arithmetic and algebra independently of geometry. Of interest to us is a work by *Alḳalsâdî* of Andalusia or of Granada, who died in 1486 or 1477.[2] His book was entitled *Raising of the Veil of the Science of Gubâr.* The word "gubâr" meant originally "dust," and stands here for written arithmetic with numerals, in contrast to mental arithmetic. In addition, subtraction, and multiplication, the result is written *above* the other figures. The square root is indicated by $\sqrt{\ }$, the initial letter of the word *dschiḏr*, meaning "root," particularly "square root." Thus, $\sqrt{48} = \overset{\backsim}{48}$. The proportion $7 : 12 = 84 : x$ was written $\backsim \therefore 84 \therefore 12 \therefore 7$, the symbol for the unknown being here probably imagined to be the initial letter of the word *dschahala*, "not to know." Observe that the Arabic writing is from right to left. In algebra proper, the unknown was expressed by the words

[1] Consult CANTOR, I., 721, 725, 781, 776–778; HANKEL, pp. 274–280.
[2] CANTOR, I., 810–817.

schai or *dschidr*, for which Alḳalsâdî uses the abbreviations $x = ش$, $x^2 = ـــ$, ل for equality. Thus, he writes $3x^2 = 12x + 63$ in this way

$$63 \overset{ش}{12} \text{ ل } \overset{ـــ}{3}.$$

This symbolism probably began to be developed among the Western Arabs at least as early as the time of Ibn Albannâ (born 1252 or 1257). It is of interest, when we remember that in the Latin translations made by Europeans this symbolism was imitated.

Europe during the Middle Ages

The barbaric nations, which from the swamps and forests of the North and from the Ural Mountains swept down upon Europe and destroyed the Roman Empire, were slower than the Mohammedans in acquiring the intellectual treasures and the civilization of antiquity. The first traces of mathematical knowledge in the Occident point to a Roman origin.

Introduction of Roman Arithmetic. — After Boethius and Cassiodorius, mathematical activity in Italy died out. About one century later, *Isidorus* (570–636), bishop of Seville, in Spain, wrote an encyclopædia, entitled *Origines*. It was modelled after the Roman encyclopædias of Martinus Capella and of Cassiodorius. Part of it is taken up with the quadruvium. He gives definitions of technical terms and their etymologies; but does not describe the modes of computation then in vogue. He divides numbers into odd and even, speaks of perfect and excessive numbers, etc., and finally bursts out in admiration of number, as follows: "Take away number from all things, and everything goes to destruction."[1] After Isidorus comes another century of complete darkness; then

[1] Cantor, I., 824.

appears the English monk, *Bede the Venerable* (672–735). His works contain treatises on the *Computus*, or the computation of Easter-time, and on finger-reckoning. It appears that a finger-symbolism was then widely used for calculation. The correct determination of the time of Easter was a problem which in those days greatly agitated the church. The desirability of having at least one monk in each monastery who could determine the date of religious festivals appears to have been the greatest incentive then existing toward the study of arithmetic. "The computation of Easter-time," says Cantor, "the real central point of time-computation, is founded by Bede as by Cassiodorius and others, upon the coincidence, once every nineteen years, of solar and lunar time, and makes no immoderate demands upon the arithmetical knowledge of the pupil who aims to solve simply this problem."[1] It is not surprising that Bede has but little to say about fractions. In one place he mentions the Roman duodecimal division into ounces.

The year in which Bede died is the year in which the next prominent thinker, *Alcuin* (735–804) was born. Educated in Ireland, he afterwards, at the court of Charlemagne, directed the progress of education in the great Frankish Empire. In the schools founded by him at the monasteries were taught the psalms, writing, singing, computation (*computus*), and grammar. As the determination of Easter could be of no particular interest or value to boys, the word *computus* probably refers to computation in general. We are ignorant of the modes of reckoning then employed. It is not probable that Alcuin was familiar with the abacus or the apices of Boethius. He belonged to that long list of scholars of the Middle Ages and of the Renaissance who dragged the theory of numbers into

[1] CANTOR, I., 831.

theology. For instance, the number of beings created by God, who created all things well, is six, because six is a perfect number (being equal to the sum of its divisors 1, 2, 3); but 8 is a defective number, since its divisors $1 + 2 + 4 < 8$, and, for that reason, the second origin of mankind emanated from the number 8, which is the number of souls said to have been in Noah's Ark.

There is a collection of "Problems for Quickening the Mind" which is certainly as old as 1000 A.D., and possibly older. Cantor is of the opinion that it was written much earlier, and by Alcuin. Among the arithmetical problems of this collection are the fountain-problems which we have encountered in Heron, in the Greek anthology, and among the Hindus. Problem No. 26 reads: A dog chasing a rabbit, which has a start of 150 feet, jumps 9 feet every time the rabbit jumps 7. To determine in how many leaps the dog overtakes the rabbit, 150 is to be divided by 2. The 35th problem is as follows: A dying man wills that if his wife, being with child, gives birth to a son, the son shall inherit $\frac{3}{4}$ and the widow $\frac{1}{4}$ of the property; but if a daughter is born, she shall inherit $\frac{7}{12}$ and the widow $\frac{5}{12}$ of the property. How is the property to be divided if both a son and a daughter are born? This problem is of interest because by its close resemblance to a Roman problem it unmistakably betrays its Roman origin. However, its solution, given in the collection, is different from the Roman solution, and is quite erroneous. Some of the problems are geometrical, others are merely puzzles, such as the one of the wolf, goat, and cabbage-head, which we shall mention again. The collector of these problems evidently aimed to entertain and please his readers. It has been remarked that the proneness to propound jocular questions is truly Anglo-Saxon, and that Alcuin was particularly noted in this respect. Of interest is the title which the collection

bears: "Problems for Quickening the Mind." Do not these words bear testimony to the fact that even in the darkness of the Middle Ages the mind-developing power of mathematics was recognized? Plato's famous inscription over the entrance of his academy is frequently quoted; here we have the less weighty, but significant testimony of a people hardly yet awakened from intellectual slumber.

During the wars and confusion which followed the fall of the empire of Charlemagne, scientific pursuits were abandoned, but they were revived again in the tenth century, principally through the influence of one man, — *Gerbert.* He was born in Aurillac in Auvergne, received a monastic education, and engaged in study, chiefly of mathematics, in Spain. He became bishop at Rheims, then at Ravenna, and finally was made Pope under the name of Sylvester II. He died in 1003, after a life involved in many political and ecclesiastical quarrels.

Gerbert made a careful study of the writings of Boethius, and published two arithmetical works, — *Rule of Computation on the Abacus,* and *A Small Book on the Division of Numbers.* Now for the first time do we get some insight into methods of computation. Gerbert used the abacus, which was probably unknown to Alcuin. In his younger days Gerbert taught school at Rheims — the trivium and quadruvium being the subjects of instruction — and one of his pupils tells us that Gerbert ordered from his shield-maker a leathern calculating board, which was divided into 27 columns, and that counters of horn were prepared, upon which the first nine numerals (apices) were marked. *Bernelinus,* a pupil of Gerbert, describes the abacus as consisting of a smooth board upon which geometricians were accustomed to strew blue sand, and then to draw their diagrams. For arithmetical purposes the board was divided into 30 columns, of which three were reserved for fractions, while the remaining 27 were divided into groups

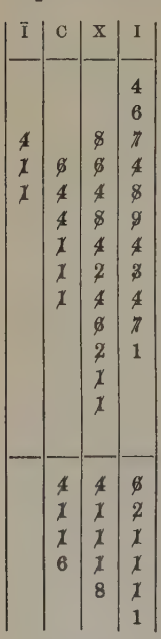
with three columns in each. Every group of the columns was marked respectively by the letters *C* (centum, 100), *D* (decem, 10), and *S* (singularis) or *M* (monas). Bernelinus gives the nine numerals used (the apices of Boethius), and then remarks that the Greek letters may be used in their place.[1] By the use of these columns any number can be written without introducing the zero, and all operations in arithmetic can be performed. Indeed, the processes of addition, subtraction, and multiplication, employed by the abacists, agreed substantially with those of to-day. The adjoining figure shows the multiplication of 4600 by 23.[2] The process is as follows: $3 \cdot 6 = 18$; $3 \times 4 = 12$; $2 \cdot 6 = 12$; $2 \times 4 = 8$; $1 + 2 + 2 = 5$; remove the 1, 2, 2, and put down 5; $1 + 1 + 8 = 10$; remove 1, 1, 8, and put down 1 in the column next to the left. Hence, the sum 105800. If counters were used, then our crossing out of digits (for example, of the digits 1, 2, 2 in the fourth column) must be imagined to represent the removal of the counters 1, 2, 2, and the putting of a counter marked 5 in their place. If the numbers were written on sand, then the numbers 1, 2, 2 were erased and 5 written instead.

C	X	I	C	X	I
		4	6		
1	1	1	8		
	1	2			
	8	2			
		5			
				2	3

The process of division was entirely different from the modern. So difficult has this operation appeared that the concept of a *quotient* may almost be said to be foreign to antiquity. Gerbert gave rules for division which apparently were framed to satisfy the following three conditions: (1) The use of the multiplication table shall be restricted as far as possible; at least, it shall never be required to multiply mentally a number of two digits by another of one digit;

[1] Cantor, I., 881. [2] Friedlein, p. 106.

(2) Subtractions shall be avoided as much as possible and replaced by additions; (3) The operation shall proceed in a purely mechanical way, without requiring trials.[1] That it should be necessary to make such conditions, will perhaps not seem so strange, if we recollect that monks of the Middle Ages did not attend school during childhood and learn the multiplication table while the memory was fresh. Gerbert's rules for division are the oldest extant. They are so brief as to be very obscure to the uninitiated, but were probably intended to aid the memory by calling to mind the successive steps of the process. In later manuscripts they are stated more fully. We illustrate this division by the example[2] $4087 \div 6 = 681$. The process is a kind of "complementary division." Beginnings of this mode of procedure are found among the Romans, but so far as known it was never used by the Hindus or Arabs. It is called "complementary" because, in our example, for instance, not 6, but $10 - 6$ or 4 is the number operated with. The *rationale* of the process may, perhaps, be seen from this partial explanation: $4000 \div 10 = 400$, write this below as part of the quotient. But 10 is too large a divisor; to rectify the error, add $4.400 = 1600$. Then $1000 \div 10 = 100$, write this below as part of the quotient; to rectify this new error, add $4.100 = 400$. Then $600 + 400 = 1000$. Divide $1000 \div 10$, and so on. It will be observed that in complementary division, like this, it was not necessary to



[1] Hankel, p. 323.

[2] Quoted by Friedlein, p. 109. The mechanism of the division is as follows: Write down the dividend 4087 and above it the divisor 6.

know the multiplication table above the 5's.[1] To a modern computor it would seem as though the above process of division were about as complicated as human ingenuity could make it. No wonder that it was said of Gerbert that he gave rules for division which were hardly understood by the most painstaking abacists; no wonder that the Arabic method of division, when first introduced into Europe, was called the "golden division" (divisio aurea), but the one on the abacus the "iron division" (divisio ferrea).

The question has been asked, whence did Gerbert get his abacus and his complementary division? The abacus was probably derived from the works of Boethius, but the complementary division is nowhere found in its developed form before the time of Gerbert. Was it mainly his invention? From one of his letters it appears that he had studied a paper on multiplication and division by "Joseph Sapiens," but modern research has as yet revealed nothing regarding this man or his writings.[2]

Above the 6 write 4, which is the difference between 10 and 6. Multiply this difference 4 into 4 in the column $\bar{I}$ and move the product 16 to the right by one column; erase the 4 in column $\bar{I}$ and write it in column C, below the lower horizontal line, as part of the quotient. Multiply the 1 in $\bar{I}$ by 4, write the product 4 in column C; erase the 1 and write it below, one column to the right. Add the numbers in C, $6 + 4 = 10$, and write 1 in $\bar{I}$. Then proceed as before: $1.4 = 4$, write it in C, and write 1 below. $4.4 = 16$ in C and X, 4 below in X; $1.4 = 4$ in X, 1 below; $4 + 6 + 8 = 18$ in C and X; $1.4 = 4$ in X, 1 below; $4 + 8 = 12$ in C and X; $1.4 = 4$ in X, 1 below; $2 + 4 = 6$ in X, $6.4 = 24$ in X and I, 6 below; $2.4 = 8$ in I, 2 below; $8 + 4 + 7 = 19$ in X and I; $1.4 = 4$ in I, 1 below; $9 + 4 = 13$ in X and I; $1.4 = 4$ in I, 1 below; $3 + 4 = 7$. Dividing 7 by 6 goes 1 and leaves 1. Write the 1 in I above and also below. Add the digits in the columns below and the sum 681 is the answer sought, *i.e.* $4087 \div 6 = 681$, leaving the remainder 1.

[1] For additional examples of complementary division see FRIEDLEIN, pp. 109–124; GÜNTHER, *Math. Unterr. im d. Mittela.*, pp. 102–106.

[2] Consult H. WEISSENBORN, *Einführung der jetzigen Ziffern in Europa*, Berlin, 1892.

In course of the next five centuries the instruments for abacal computation were considerably modified. Not only did the computing tables, strewn with sand, disappear, but also Gerbert's abacus with vertical columns and marked counters (apices). In their place there was used a calculating board with lines drawn *horizontally* (from left to right) and with counters all alike and unmarked. Its use is explained in the first printed arithmetics, and will be described under Recorde. The new instrument was employed in Germany, France, England, but not in Italy.[1]

Translation of Arabic Manuscripts. — Among the translations which were made in the period beginning with the twelfth century is the arithmetic of Alchwarizmî (probably translated by Athelard of Bath), the algebra of Alchwarizmî (by Gerard of Cremona in Lombardy) and the astronomy of Al Battânî (by Plato of Tivoli). John of Seville wrote a *liber alghoarismi*, compiled by him from Arabic authors. Thus Arabic arithmetic and algebra acquired a foothold in Europe. Arabic or rather Hindu methods of computation, with the zero and the principle of local value, began to displace the abacal modes of computation. But the victory of the new over the old was not immediate. The struggle between the two schools of arithmeticians, the old *abacistic school* and the new *algoristic school*, was incredibly long. The works issued by the two schools possess most striking differences, from which it would seem clear that the two parties drew from independent sources, and yet it is argued by some that Gerbert got his apices and his arithmetical knowledge, not from Boethius, but from the Arabs in Spain, and that part or the whole of the geometry of Boethius is a forgery, dating from the time of Gerbert. If this were the case, then we should

[1] See CANTOR, Vol. II., 2nd Edition, 1900, pp. 216, 217; see also GERHARDT, *Geschichte der Mathematik in Deutschland*, München, 1877, p. 29.

expect the writings of Gerbert to betray Arabic sources, as do those of John of Seville. But no points of resemblance are found. Gerbert could not have learned from the Arabs the use of the abacus, because we possess no reliable evidence that the Arabs ever used it. The contrast between algorists and abacists consists in this, that unlike the latter, the former mention the Hindus, use the term *algorism*, calculate with the zero, and do not employ the abacus. The former teach the extraction of roots, the abacists do not; the algorists teach sexagesimal fractions used by the Arabs, while the abacists employ the duodecimals of the Romans.

The First Awakening. — Towards the close of the twelfth century there arose in Italy a man of genuine mathematical power. He was not a monk, like Bede, Alcuin, and Gerbert, but a business man, whose leisure hours were given to mathematical study. To *Leonardo of Pisa*, also called *Fibonacci*, or *Fibonaci*, we owe the first renaissance of mathematics on Christian soil. When a boy, Leonardo was taught the use of the abacus. In later years, during his extensive travels in Egypt, Syria, Greece, and Sicily, he became familiar with various modes of computation. Of the several processes he found the Hindu unquestionably the best. After his return home, he published in 1202 a Latin work, the *liber abaci*. A second edition appeared in 1828. While this book contains pretty much the entire arithmetical and algebraical knowledge of the Arabs, it demonstrates its author to be more than a mere compiler or slavish imitator. The *liber abaci* begins thus: "The nine figures of the Hindus are 9, 8, 7, 6, 5, 4, 3, 2, 1. With these nine figures and with this sign, 0, which in Arabic is called *sifr*, any number may be written." The Arabic *sifr* (*sifra* = empty) passed into the Latin *zephirum* and the English *cipher*. If it be remembered that the Arabs wrote from *right to left*, it becomes evident how Leonardo,

in the above quotation, came to write the digits in descending rather than ascending order; it is plain also how he came to write $\frac{1}{2}182$ instead of $182\frac{1}{2}$. The *liber abaci* is the earliest work known to contain a recurring series.[1] Interesting is the following problem of the seven old women, because it is given (in somewhat different form) by Ahmes, 3000 years earlier: Seven old women go to Rome, each woman has seven mules, each mule carries seven sacks, each sack contains seven loaves, with each loaf are seven knives, each knife rests in seven sheaths. What is the sum total of all named? *Ans.* 137256.[2] Leonardo's treatise was for centuries the storehouse from which authors drew material for their arithmetical and algebraical books. Leonardo's algebra was purely "rhetorical"; that is, devoid of all algebraic symbolism.

Leonardo's fame spread over Italy, and Emperor Frederick II. of Hohenstaufen desired to meet him. The presentation of the celebrated algebraist to the great patron of learning was accompanied by a famous scientific tournament. John of Palermo, an imperial notary, proposed several problems which Leonardo solved promptly. The first was to find the number x, such that x^2+5 and x^2-5 are each square numbers. The answer is $x=3\frac{5}{12}$; for $(3\frac{5}{12})^2+5=(4\frac{1}{12})^2$; $(3\frac{5}{12})^2-5=(2\frac{7}{12})^2$. The Arabs had already solved similar problems, but some parts of Leonardo's solution seem original with him. The second problem was the solution of $x^3+2x^2+10x=20$. The general algebraic solution of cubic equations was unknown at that time, but Leonardo succeeded in approximating to one of the roots. He gave $x=1^\circ 22' 7'' 4''' 33^{iv} 4^v 40^{vi}$, the answer being thus expressed in sexagesimal fractions. Converted into decimals, this value furnishes figures correct to nine places. These and other problems solved by Leonardo disclose brilliant

[1] CANTOR, II., 26. [2] CANTOR, II., 26.

talents. His geometrical writings will be touched upon later.

In Italy the Hindu numerals were readily accepted by the enlightened masses, but at first rejected by the learned circles. Italian merchants used them as early as the thirteenth century; in 1299 the Florentine merchants were forbidden the use of the Hindu numerals in bookkeeping, and ordered either to use the Roman numerals or to write numbers out in words.[1] The reason for this decree lies probably in the fact that the Hindu numerals as then employed had not yet assumed fixed, definite shapes, and the variety of forms for certain digits sometimes gave rise to ambiguity, misunderstanding, and fraud. In our own time, even, sums of money are always written out in words in case of checks or notes. Among the Italians are evidences of an early maturity of arithmetic. Says Peacock, "The Tuscans generally, and the Florentines in particular, whose city was the cradle of the literature and arts of the thirteenth and fourteenth centuries, were celebrated for their knowledge of arithmetic; the method of bookkeeping, which is called especially Italian, was invented by them; and the operations of arithmetic, which were so necessary to the proper conduct of their extensive commerce, appear to have been cultivated and improved by them with particular care; to them we are indebted . . . for the formal introduction into books of arithmetic, under distinct heads, of questions in the single and double rule of three, loss and gain, fellowship, exchange, simple interest, discount, compound interest, and so on."[2]

In Germany, France, and England, the Hindu numerals were scarcely used, until after the middle of the fifteenth century.[3] A small book on Hindu arithmetic, entitled *De arte numerandi*, called also *Algorismus*, was read, mainly in France

[1] Hankel, p. 341. [2] Peacock, p. 414. [3] See Unger, p. 14.

and Italy, for several centuries. It is usually ascribed to *John Halifax*, also called *Sacrobosco*, or *Holywood*, who was born in Yorkshire, and educated at Oxford, but who afterwards settled in Paris and taught there until his death, in 1256. The booklet contains rules without proofs and without numerical examples; it ignores fractions. It was printed in 1488 and many times later. According to De Morgan it is the "first arithmetical work ever printed in a French town (Strasbourg)."[1]

Here and there some of our modern notions were anticipated by writers of the Middle Ages. For example, *Nicole Oresme*, a bishop in Normandy (about 1323–1382), first conceived the notion of fractional powers, afterwards rediscovered by Stevin, and suggested a notation for them. Thus,[2] since $4^3 = 64$ and $\sqrt{64} = 8$, it follows that $4^{1\frac{1}{2}} = 8$. In Oresme's notation $4^{1\frac{1}{2}}$ is expressed, $\boxed{1\,p\cdot\frac{1}{2}}\ 4$, or $\boxed{\frac{p\cdot 1}{1\cdot 2}}\ 4$. Such suggestions to the contrary notwithstanding, the fact remains that the fourteenth and fifteenth centuries brought forth comparatively little in the way of original mathematical research. There were numerous writers, but their scientific efforts were vitiated by the methods of scholastic thinking.

Geometry and Trigonometry

Hindus

Our account of Hindu geometrical research will be very brief; for, in the first place, like the Egyptians and Romans, the Hindus never possessed a *science* of geometry; in the second place, unlike the Egyptians and Romans, they do not

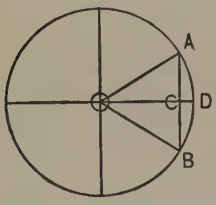
[1] See *Biblioth. Mathem.*, 1894, pp. 73–78; also 1895, pp. 36–37; Cantor, II., pp. 88–91; "De arte numerandi" was reprinted last by J. O. Halliwell, in *Rara Mathematica*, 1839.

[2] Cantor, II., 133.

figure in geometrical matters as the teachers of other nations. Their early geometry was original, their later geometry came from Greece and China. Brahmagupta gives the "Heronic Formula" for the area of a triangle. He also gives the proposition of Ptolemæus, that the product of the diagonals of a quadrilateral is equal to the sum of the product of the opposite sides, and adds new theorems of his own on quadrilaterals inscribed in a circle. The calculation of areas forms the chief part of Hindu geometry. Aryabhatta gives $\pi = \frac{3\,1416}{10000}$. Interesting is Bhaskara's proof of the theorem of the right triangle. He draws a right triangle four times in the square of the hypotenuse, so that in the middle there remains a square whose side equals the difference between the two sides of the right triangle. Arranging this small square and the four triangles in a different way, they are seen, together, to make up the sum of the squares of the two sides. "Behold," says Bhaskara, without adding another word of explanation. Rigid forms of demonstration are unusual with the Hindus. This proof is given much earlier by a Chinese writer. Pythagoras' proof may have been like this. In another place Bhaskara gives a second demonstration of this theorem by drawing from the vertex of the right angle a perpendicular to the hypotenuse, and then suitably manipulating the proportions yielded by the similar triangles. This proof was unknown in Europe until it was rediscovered by Wallis.

More successful were the Hindus in the cultivation of trigonometry. As with the Greeks, so with them, it was valued merely as a tool in astronomical research. Like the Babylonians and Greeks, they divide the circle into 360 degrees and 21,600 minutes. Taking $\pi = 3.1416$, and $2\pi r = 21{,}600$,

they got $r = 3438$; that is, the radius contained nearly 3438 of these circular parts. This step was not Grecian. Some Greek mathematicians would have had scruples about measuring *a straight line* by a part of a *curve*. With Ptolemy the division of the radius into sexagesimal parts was independent of the division of the circumference; no *common* unit of measure was selected. The Hindus divided each quadrant into 24 equal parts, so that each of these parts contained 225 out of the 21,600 units. A vital feature of Hindu trigonometry is that they did not, like the Greeks, reckon with the whole chord of double a given arc, but with the *sine* of the arc (*i.e.* half the chord of double the arc) and with the *versed sine* of the arc. The entire chord AB was called by the Brahmins *jyâ* or *jîva*, words which meant also the cord of a hunter's bow. For AC, or half the chord, they used the words *jyârdha* or *ardhajyâ*, but the names of the whole chord were also used for brevity. It is interesting to trace the history of these words. The Arabs transliterated *jivâ* or *jîva* into *dschîba*. For this they afterwards used the word *dschaib*, of nearly the same form, meaning "bosom." This, in turn, was translated into Latin, as *sinus*, by Plato of Tivoli. Thus arose the word *sine* in trigonometry. For "versed sine," the Hindus used the term *utkramajyâ*, for "cosine," *kotijyâ*.[1] Observe that the Hindus used three of our trigonometric functions, while the Greeks considered only the chord.



The Hindus computed a table of sines by a theoretically simple method. The sine of 90° was equal to the radius, or 3438; the chord of an arc AB of 60° was also 3438, therefore half this chord AC, or the *sine* of 30°, was 1719. Applying the

[1] Cantor, I., 658, 736.

formula $\sin^2 a + \cos^2 a = r^2$, and observing that $\sin 45° = \cos 45°$, they obtained $\sin 45° = \sqrt{\frac{r^2}{2}} = 2431$. Substituting for $\cos a$ its equal $\sin (90 - a)$, and making $a = 60°$, they obtained

$$\sin 60° = \tfrac{1}{2}\sqrt{3}\, r^2 = 2978.$$

With the sines of 90, 60, 45 as starting-points, they reckoned the sines of half the angles by the formula $\text{versin } 2a = 2 \sin^2 a$, thus obtaining the sines of 22° 30′, 15°, 11° 15′, 7° 30′, 3° 45′. They now figured out the sines of the complements of these angles, namely, the sines of 86° 15′, 82° 30′, 78° 45′, 75°, 67° 30′; then they calculated the sines of half these angles; thereof their complements, and so on. By this very simple process they got the sines of all the angles at intervals of 3° 45′.[1] No Indian treatise on the trigonometry of the triangle is extant. In astronomical works, there are given solutions of plane and spherical right triangles. Scalene triangles were divided up into right triangles, whereby all ordinary computations could be carried out. As the table of sines gave the values for angles at intervals of $3\frac{3}{4}$ degrees, the sines of intervening angles had to be found by interpolation. Astronomical observations and computations possessed only a passable degree of accuracy.[2]

Arabs

The Arabs added hardly anything to the ancient stock of geometrical knowledge. Yet they play an all-important rôle in mathematical history; they were the custodians of Greek and Oriental science, which, in due time, they transmitted to

[1] A. Arneth, *Die Geschichte der reinen Mathematik*, Stuttgart, 1852, pp. 172, 173. This work we shall cite as Arneth. See, also, Hankel, p. 217.

[2] Arneth, p. 174.

the Occident. The starting-point for all geometric study among the Arabs was the *Elements* of Euclid. Over and over again was this great work translated by them into the Arabic tongue. Imagine the difficulties encountered in such a translation. Here we see a people, just emerged from barbarism, untrained in mathematical thinking, and with limited facilities for the accurate study of languages. Where was to be found the man, who, without the aid of grammars and dictionaries, had become versed in both Greek and Arabic, and was at the same time a mathematician? How could highly refined scientific thought be conveyed to undeveloped minds by an undeveloped language? Certainly it is not strange that several successive efforts at translation had to be made, each translator resting upon the shoulders of his predecessor.

Arabic rulers wisely enlisted the aid of Greek scholars. In Syria the sciences, especially philosophy and medicine, were cultivated by Greek Christians. Celebrated were the schools at Antioch and Emesa, and the Nestorian school at Edessa. After the sack and ruin of Alexandria, in 640, they became the chief repositories in the East of Greek learning. Euclid's *Elements* were translated into Syriac. From Syria Greek Christians were called to Bagdad, the Mohammedan capital. During the time of the Caliph *Hârûn ar-Raschîd* (786–809) was made the first Arabic translation of Ptolemy's *Almagest;* also of Euclid's *Elements* (first six books) by *Haddschâdsch ibn Jûsuf ibn Matar.*[1] He made a second translation under the Caliph Al Mamûn (813–833). This caliph secured as a condition, in a treaty of peace with the emperor in Constanti-

[1] Cantor, I., 702; *Biblioth. Mathem.*, 1892, p. 65. An account of translators and commentators on Euclid was given by *Ibn Abî Ja'kûb an-Nadîm* in his *Fihrist,* an important bibliographical work published in Arabic in 987. See a German translation by H. Suter, in the *Zeitschr. für Math. u. Phys.*, 1892, Supplement, pp. 3–87.

nople, a large number of Greek manuscripts, which he ordered translated into Arabic. Euclid's *Elements* and the *Sphere and Cylinder* of Archimedes were translated by *Abû Ja'ḳûb Isḥâḳ ibn Ḥunain*, under the supervision of his father *Hunain ibn Isḥâḳ*.[1] These renderings were unsatisfactory; the translators, though good philologians, were poor mathematicians. At this time there were added to the thirteen books of the *Elements* the fourteenth, by Hypsicles (?), and the fifteenth by Damascius (?). It remained for *Tâbit ibn Ḳurrah* (836–901) to bring forth an Arabic Euclid satisfying every need. Among other important translations into Arabic were the mathematical works of Apollonius, Archimedes, Heron, and Diophantus. Thus, in course of one century, the Arabs gained access to the vast treasures of Greek science.

A later and important Arabic edition of Euclid's *Elements* was that of the gifted *Nasîr Eddîn* (1201–1274), a Persian astronomer who persuaded his patron Hûlâgû to build him and his associates a large observatory at Maraga. He tried his skill at a proof of the parallel-postulate. In all such attempts, some new assumption is made which is equivalent to the thing to be proved. Thus Nasîr Eddîn assumes that if AB is $\perp$ to CD at C, and if another straight line EDF makes the angle EDC acute, then the perpendiculars to AB, comprehended between AB and EF, and drawn on the side of CD toward E, are shorter and shorter, the further they are from CD.[2] It is difficult to see how in any case this can be otherwise, unless one looks with the eyes of Lobatchewsky or Bolyai. Nasîr Eddîn's "proof" had some influence on the later development of the theory of parallels. His edition of Euclid was printed in Arabic at Rome in 1594 and his "proof" was brought out

[1] CANTOR, I., 703.

[2] The proof is given by KÄSTNER, I., 375–381 and in part, in *Biblioth. Mathem.*, 1892, p. 5.

in Latin translation by Wallis in 1651.[1] Of interest is a new proof of the Pythagorean Theorem which Nasîr Eddîn adds to the Euclidean proof.[2] An earlier demonstration, for the special case of an isosceles right triangle,[3] is given by *Muḥammed ibn Mûsâ Alchwarizmî* who lived during the reign of Caliph Al Mamûn, in the early part of the ninth century. Alchwarizmî's meagre treatment of geometry, as contained in his work on Algebra, is the earliest Arabic effort in this science. It bears unmistakable evidence of Hindu influences. Besides the value $\pi = 3\frac{1}{7}$, it contains also the Hindu values $\pi = \sqrt{10}$ and $\pi = \frac{62832}{20000}$. In later Arabic works, Hindu geometry hardly ever shows itself; Greek geometry held undisputed sway. In a book by the sons of *Mûsâ ibn Schâkir* (who in his youth was a robber) is given the Heronic Formula for the area of a triangle. A neat piece of research is displayed in the "geometric constructions" by *Abû'l Wafâ* (940–998), a native of Buzshan in Chorassan. He improved the theory of draughting by showing how to construct the corners of the regular polyedrons on the circumscribed sphere. Here, for the first time, appears the condition which afterwards became very famous in the Occident, that the construction be effected with a single opening of the compasses.

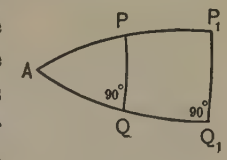
The best original work done by the Arabs in mathematics

[1] Wallis, *Opera*, II., 669–673.

[2] See H. Suter, in *Biblioth. Mathem.*, 1892, pp. 3 and 4. In Hoffmann's and Wipper's collections of proofs for this theorem, Nasîr Eddîn's proof is given without any reference to him.

[3] Some Arabic writers, Behâ Eddîn for instance, called the Pythagorean Theorem the "figure of the bride." This romantic appellation originated probably in a mistranslation of the Greek word *νύμφη*, applied to the theorem by a Byzantine writer of the thirteenth century. This Greek word admits of two meanings, "bride" and "winged insect." The figure of a right triangle with its three squares suggests an insect, but Behâ Eddîn apparently translated the word as "bride." See Paul Tannery, in *L'Intermediaire des Mathématiciens*, 1894, T. I., p. 254.

is the geometric solution of cubic equations and the development of trigonometry. As early as 773 Caliph Almansur came into possession of the Hindu table of sines, probably taken from Brahmagupta's *Siddhânta*. The Arabs called the table the *Sindhind* and held it in high authority. They also came into early possession of Ptolemy's *Almagest* and of other Greek astronomical works. *Muhammed ibn Mûsâ Alchwarizmî* was engaged by Caliph Al Mamûn in making extracts from the *Sindhind*, in revising the tables of Ptolemy, in taking observations at Bagdad and Damascus, and in measuring the degree of the earth's meridian. Remarkable is the derivation, by Arabic authors, of formulæ in spherical trigonometry, not from the "rule of six quantities of Menelaus," as previously, but from the "rule of four quantities." This is: If PP_1 and QQ_1 be two arcs of great circles intersecting in A, and if PQ and P_1Q_1 be arcs of great circles drawn perpendicular to QQ_1, then we have the proportion



$$\sin AP : \sin PQ = \sin AP_1 : \sin P_1Q_1.$$

This departure from the time-honoured procedure adopted by Ptolemy was formerly attributed to *Dschâbir ibn Aflah*, but recent study of Arabic manuscripts indicates that the transition from the "rule of six quantities" to the "rule of four quantities" was possibly effected already by *Tâbit ibn Ḳurrah*[1] (836–901), the change being adopted by other writers who preceded Dschâbir ibn Aflah.[2]

Foremost among the astronomers of the ninth century ranked

[1] H. Suter, in *Biblioth. Mathem.*, 1893, p. 7.

[2] For the mode of deriving formulæ for spherical right triangles, according to Ptolemy, also according to Dschâbir ibn Aflah and his Arabic predecessors, see Hankel, pp. 285–287; Cantor, I., 794.

Al Battânî, called *Albategnius* by the Latins. Battan in Syria was his birth-place. His work, *De scientia stellarum*, was translated into Latin by *Plato Tiburtinus*, in the twelfth century. In this translation the Arabic word *dschîba*, from the Sanskrit *jîva*, is said to have been rendered by the word *sinus;* hence the origin of "sine." Though a diligent student of Ptolemy, Al Battânî did not follow him altogether. He took an important step for the better, when he introduced the Indian "sine" or *half* the chord, in place of the *whole* chord of Ptolemy. Another improvement on Greek trigonometry made by the Arabs points likewise to Indian influences: Operations and propositions treated by the Greeks geometrically, are expressed by the Arabs algebraically. Thus Al Battânî at once gets from an equation $\frac{\sin\theta}{\cos\theta} = D$, the value of θ by means of $\sin\theta = D \div \sqrt{1 + D^2}$, a process unknown to Greek antiquity.[1] To the formulæ known to Ptolemy he adds an important one of his own for oblique-angled spherical triangles; namely, $\cos a = \cos b \cos c + \sin b \sin c \cos A$.

Important are the researches of *Abû'l Wafâ*. He invented a method for computing tables of sines which gives the sine of half a degree correct to nine decimal places.[2] He and Al-Battânî, in the study of shadow-triangles cast by sun-dials, introduced the trigonometric functions *tangent* and *cotangent;* he used also the *secant* and *cosecant*. An important change in method was inaugurated by *Dschâbir ibn Aflah* of Seville in Spain (in the second half of the eleventh century) and by *Nasîr Eddîn* (1201–1274) in distant Persia. In the works of the last two authors we find for the first time trigonometry developed as a part of pure mathematics, independently of astronomy.

[1] Cantor, I., p. 737.

[2] Consult Cantor, I., 746–748.

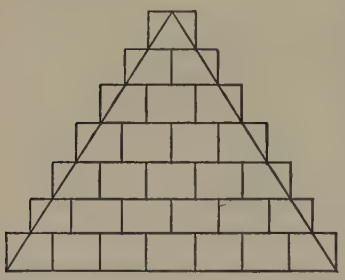
We do not desire to go into greater detail, but would emphasize the fact that *Nasîr Eddîn* in the far East, during a temporary cessation of military conquests by Tartar rulers, developed both plane and spherical trigonometry to a very remarkable degree. Suter[1] enthusiastically asks, what would have remained for European scholars of the fifteenth century to do in trigonometry, had they known of these researches? Or were some of them, perhaps, aware of these investigations? To this question we can, as yet, give no final answer.

Europe during the Middle Ages

Introduction of Roman Geometry. — Before the introduction of Arabic learning into Europe, the knowledge of geometry in the Occident cannot be said to have exceeded that of the Egyptians in 600 B.C. The monks of the Middle Ages did not go much beyond the definitions of the triangle, quadrangle, circle, pyramid, and cone (as given in the Roman encyclopedia of the Carthaginian, Martianus Capella), and the simple rules of mensuration. In *Alcuin's* "Problems for Quickening the Mind," the areas of triangular and quadrangular pieces of land are found by the same formulæ of approximation as those used by the Egyptians and given by Boethius in his geometry: The rectangle equals the product of half the sums of the opposite sides; the triangle equals the product of half the sum of two sides and half the third side. After Alcuin, the great mathematical light of Europe was Gerbert (died 1003). In Mantua he found the geometry of Boethius and studied it zealously. It is usually believed that Gerbert himself was the author of a geometry. This contains nothing

[1] For further particulars consult his article in *Biblioth. Mathem.*, 1893, pp. 1–8.

more than the geometry of Boethius, but the fact that occasional errors in Boethius are herein corrected shows that the author had mastered the subject.[1] "The earliest mathematical paper of the Middle Ages which deserves this name is a letter of Gerbert to Adalbold, bishop of Utrecht,"[2] in which is explained the reason why the area of a triangle, obtained "geometrically" by taking the product of the base by its altitude, differs from the area calculated "arithmetically," according to the formula $\frac{1}{2}a(a+1)$, used by surveyors, where a stands for a side of an equilateral triangle. Gerbert gives the correct explanation, that in the latter formula all the small rectangles, into which the triangle is supposed to be divided, are counted in wholly, even though parts of them project beyond it.



Translation of Arabic Manuscripts. — The beginning of the twelfth century was one of great intellectual unrest. Philosophers longed to know more of Aristotle than could be learned through the writings of Boethius; mathematicians craved a profounder mathematical knowledge. Greek texts were not at hand; so the Europeans turned to the Mohammedans for instruction; at that time the Arabs were the great scholars of the world. We read of an English monk, *Athelard of Bath*, who travelled extensively in Asia Minor, Egypt, and Spain, braving a thousand perils, that he might acquire the language and science of the Mohammedans. "The Moorish

[1] For description of its contents see CANTOR, I., 861–867; S. GÜNTHER, *Geschichte des Mathematischen Unterrichts im deutchen Mittelalter*, Berlin, 1887, pp. 115–120.

[2] HANKEL, p. 314.

universities of Cordova and Seville and Granada were dangerous resorts for Christians." He made what is probably the earliest translation from the Arabic into Latin of Euclid's *Elements*,[1] in 1120. He translated also the astronomical tables of Muḥammed ibn Mûsâ Alchwarizmî. In his translation of Euclid from the Arabic there is ground for suspicion that Athelard was aided by a previous Latin translation.[2]

All important Greek mathematical works were translated from the Arabic. *Gerard of Cremona* in Lombardy went to Toledo and there in 1175 translated the *Almagest*. We are told that he translated into Latin 70 works, embracing the 15 books of Euclid, Euclid's *Data*, the *Sphærica* of Theodosius, and a work of Menelaus. A new translation of Euclid's

[1] We are surprised to read in the *Dictionary of National Biography* [Leslie Stephen's] that "it has not yet been determined whether the translation of Euclid's *Elements* . . . was made from the Arabic version or from the original." To our knowledge no mathematical historian now doubts that the translation was made from the Arabic or suspects that Athelard used the Greek text. See CANTOR, I., 713, 906; II., 100. HANKEL, p. 335; S. GÜNTHER, *Math. Unt. im d. Mittelalt.*, pp. 147–149; W. W. R. BALL, 1893, p. 170; GOW, p. 206; H. SUTER, *Gesch. d. Math.*, 1., 146; HOEFER, *Histoire des Mathématiques*, 1879, p. 321. Remarkable is the fact that MARIE, in his 12-volume history of mathematics, not even mentions Athelard. He says that Campanus "a donné des *Éléments* d'Euclide la première traduction qu'on ait eue en Europe." MARIE, II., p. 158.

[2] CANTOR, II., 100–102. In a geometrical manuscript in the British Museum it is said that geometry was invented in Egypt by Eucleides. This verse is appended:

> "Thys craft com ynto England, as y ghow say,
> Yn tyme of good Kyng Adelstones day."

See HALLIWELL'S *Rara Mathematica*, London, 1841, p. 56, etc. As King Athelstan lived about 200 years before Athelard, it would seem that a Latin Euclid (perhaps only the fragments given by Boethius) was known in England long before Athelard.

Elements was made, about 1260, by *Giovanni Campano* (latinized form, *Campanus*) of Novara in Italy. It displaced the earlier ones and formed the basis of the printed editions.

The First Awakening.—The central figure in mathematical history of this period is the gifted *Leonardo of Pisa* (1175–?). His main researches are in algebra, but his *Practica Geometriæ*, published in 1220, is a work disclosing skill and geometric rigour. The writings of Euclid and of some other Greek masters were known to him, either directly from Arabic manuscripts or from the translations made by his countrymen, Gerard of Cremona and Plato of Tivoli. Leonardo gives elegant demonstrations of the "Heronic Formula" and of the theorem that the medians of a triangle meet in a point (known to Archimedes, but not proved by him). He also gives the theorem that the square of the diagonal of a rectangular parallelopiped is equal to the sum of the three squares of its sides.[1] Algebraically are solved problems like this: To inscribe in an equilateral triangle a square resting upon the base of the triangle.

A geometrical work similar to Leonardo's in Italy was brought out in Germany about the same time by the monk *Jordanus Nemorarius*. It was entitled *De triangulis*, and was printed by Curtze in 1887. It indicates a decided departure from Greek models, though to Euclid reference is frequently made. There is nothing to show that it was used anywhere as a text-book in schools. This work, like Leonardo's, was probably read only by the *élite*. As specimens of remarkable theorems, we give the following: If circles can be inscribed and circumscribed about an irregular polygon, then their centres do not coincide; of all inscribed triangles having a common base, the isosceles is the maximum. Jordanus accomplishes the trisection of an angle by giving a *graduated* ruler simultaneously a

[1] Cantor, II., p. 39.

rotating and a sliding motion, its final position being fixed with aid of a certain length marked on the ruler.[1] In this trisection he does not permit himself to be limited to Euclid's postulates, which allow the use simply of an unmarked ruler and a pair of compasses. He also introduces motion of parts of a figure after the manner of some Arabic authors. Such motion is foreign to Euclid's practice.[2] The same mode of trisection was given by Campanus.

Jordanus's attempted exact quadrature of the circle lowers him in our estimation. Circle-squaring now began to command the lively attention of mathematicians. Their efforts remained as futile as though they had attempted to jump into a rainbow; the moment they thought they had touched the goal, it vanished as by magic, and was as far as ever from their reach. In their excitement many of them became subject to mental illusions, and imagined that they had actually attained their aim, and were in the midst of a triumphal arch of glory, the wonder and admiration of the world.

The fourteenth and fifteenth centuries have brought forth no geometricians who equalled Leonardo of Pisa. Much was written on mathematics, and an effort put forth to digest the rich material acquired from the Arabs. No substantial contributions were made to geometry.

An English manuscript of the fourteenth century, on surveying, bears the title: *Nowe sues here a Tretis of Geometri wherby you may knowe the heghte, depnes, and the brede of mostwhat erthely thynges.*[3] The oldest French geometrical manuscript (of about 1275) is likewise anonymous. Like the English treatise, it deals with mensuration. From the study of

[1] Cantor, II., 81, gives the construction in full.

[2] For fuller extracts from the *De triangulis*, see Cantor, II., 73–79; S. Günther, *op. cit.*, 160–162.

[3] See Halliwell, *Rara Mathematica*, 56–71; Cantor, II., p. 112.

manuscripts, made to the present time, it would seem that since the thirteenth century surveying in Europe had departed from Roman models and come completely under the influence of the Græco-Arabic writers.[1] An author of considerable prominence was *Thomas Bradwardine* (1290?–1349), archbishop of Canterbury. He was educated at Merton College, Oxford, and later lectured in that university on theology, philosophy, and mathematics. His philosophic writings contain able discussions of the infinite and the infinitesimal — subjects which thenceforth came to be studied in connection with mathematics. Bradwardine wrote several mathematical treatises. A *Geometria speculativa* was printed in Paris in 1511 as the work of Bradwardinus, but has been attributed by some to a Dane, named *Petrus*, then a resident of Paris. This remarkable work enjoyed a wide popularity. It treats of the regular solids, of isoperimetrical figures in the manner of Zenodorus, and of star-polygons. The first appearance of such polygons was with Pythagoras and his school. The pentagram-star was used by the Pythagoreans as a badge or symbol of recognition, and was called by them Health.[2] We next meet such polygons in the geometry of Boethius, in the translation of Euclid from the Arabic by Athelard of Bath, and by Campanus, and in the earliest French geometric treatise, mentioned above. Bradwardine develops some geometric properties of star-polygons — their construction and angle-sum. We encounter these fascinating figures again in Regiomontanus, Kepler, and others.

In Bradwardine and a few other British scholars England proudly claims the earliest European writers on trigonometry. Their writings contain trigonometry drawn from Arabic

[1] Cantor, II., 127. [2] Gow, p. 151.

sources. John Maudith, professor at Oxford about 1340, speaks of the *umbra* ("tangent"); Bradwardine uses the terms *umbra recta* ("cotangent") and *umbra versa* ("tangent")[1] which occur in Gerard of Cremona's translation of Al-Arzakel's Toledian tables. The Hindus had introduced the *sine, versed sine, cosine;* the Arabs the *tangent, cotangent, secant, cosecant.*

Perhaps the greatest result of the introduction of Arabic learning was the establishment of universities. What was their attitude toward mathematics? At the University of Paris geometry was neglected. In 1336 a rule was introduced that no student should take a degree without attending lectures on mathematics, and from a commentary on the first six books of Euclid, dated 1536, it appears that candidates for the degree of A.M. had to take oath that they had attended lectures on these books.[2] Examinations, when held at all, probably did not extend beyond the first book, as is shown by the nickname "magister matheseos" applied to the theorem of Pythagoras, the last of the book. At Prague, founded in 1384, astronomy and applied mathematics were additional requirements. Roger Bacon, writing near the close of the thirteenth century, says that at Oxford there were few students who cared to go beyond the first three or four propositions of Euclid, and that on this account the fifth proposition was called "elefuga," that is, "flight of the wretched." We are told that this fifth proposition was later called the "pons asinorum" or "the Bridge of Asses."[3] Clavius in his Euclid, edition of 1591, says of this theorem, that beginners find it

[1] CANTOR, II., 111.

[2] HANKEL, pp. 354–359. We have consulted also H. SUTER, *Die Mathematik auf den Universitäten des Mittelalters*, Zürich, 1887; S. GÜNTHER, *Math. Unt. im d. Mittela.*, p. 199; CANTOR, II., pp. 139–142.

[3] This nickname is sometimes also given to the Pythagorean Theorem, I., 47, though usually I., 47, is called "the windmill." Read THOMAS CAMPBELL'S poem, "The Pons Asinorum."

difficult and obscure, on account of the multitude of lines and angles, to which they are not yet accustomed. These last words, no doubt, indicate the reason why geometrical study seems to have been so pitifully barren. Students with no kind of mathematical training, perhaps unable to perform the simplest arithmetical computations, began to memorize the abstract definitions and propositions of Euclid. Poor preparation and poor teaching, combined with an absence of rigorous requirements for degrees, probably explain this flight from geometry — this "elefuga." In the middle of the fifteenth century the first two books were read at Oxford.

Thus it is seen that the study of mathematics was maintained at the universities only in a half-hearted manner.

MODERN TIMES

ARITHMETIC

Its Development as a Science and Art

During the sixteenth century the human mind made an extraordinary effort to achieve its freedom from scholastic and ecclesiastical bondage. This independent and vigorous intellectual activity is reflected in the mathematical books of the time. The best arithmetical work of the fifteenth as also of the sixteenth century emanated from Italian writers,—Lucas Pacioli and Tartaglia. *Lucas Pacioli* (1445?–1514?)—also called *Lucas di Burgo, Luca Paciuolo,* or *Pacciuolus*—was a Tuscan monk who taught mathematics at Perugia, Naples, Milan, Florence, Rome, and Venice. His treatise, *Summa de Arithmetica,* 1494, contains all the knowledge of his day on arithmetic, algebra, and trigonometry, and is the first comprehensive work which appeared after the *liber abaci* of Fibonaci. The earliest arithmetic known to have been brought out in print anywhere appeared anonymously in Treviso, Italy, in 1478, and is known as the "Treviso arithmetic."

Tartaglia's real name was *Nicolo Fontana* (1499?–1557). When a boy of six, Nicolo was so badly cut by a French soldier, that he never again gained the free use of his tongue. Hence he was called Tartaglia, *i.e.* the stammerer. His widowed mother being too poor to pay his tuition at school, he

learned to read and acquired a knowledge of Latin, Greek, and mathematics without a teacher. Possessing a mind of extraordinary power, he was able to teach mathematics at an early age. He taught at Verona, Piacenza, Venice, and Brescia. It was his intention to embody his original researches in a great work, *General trattato di numeri et misure*, but at his death it was still unfinished. The first two parts were published in 1556, and treat of arithmetic. Tartaglia discusses commercial arithmetic somewhat after the manner of Pacioli, but with greater fulness and with simpler and more methodical treatment. His work contains a large number of exercises and problems so arranged as to insure the reader's mastery of one subject before proceeding to the next. Tartaglia bears constantly in mind the needs of the practical man. His description of numerical operations embraces seven different modes of multiplication and three methods of division.[1] He gives the Venetian weights and measures.

Mathematical study was fostered in Germany at the close of the fifteenth century by *Georg Purbach* and his pupil, *Regiomontanus*. The first printed German arithmetic appeared in 1482 at Bamberg. It is by *Ulrich Wagner*, a practitioner of Nürnberg. It was printed on parchment, but only fragments of one copy are now extant.[2] In 1483 the same Bamberg publishers brought out a second arithmetic, printed on paper, and covering 77 pages. The work is anonymous, but Ulrich Wagner is believed to be its author. It is worthy of remark that the earliest printed German arithmetic appeared only four years after the first printed Italian arithmetic. The Bamberg arithmetic of 1483, says Unger, bears no resemblance to previous Latin treatises, but is purely commercial. Modelled after it is the arithmetic by John Widmann, Leipzig, 1489.

[1] UNGER, p. 60. [2] UNGER, pp. 36–40.

This work has become famous, as being the earliest book in which the symbols + and − have been found. They occur in connection with problems worked by "false position." Widmann says, "what is −, that is minus; what is +, that is more." The words "minus" and "more," or "plus," occur long before Widmann's time in the works of Leonardo of Pisa, who uses them in connection with the method of false position in the sense of "positive error" and "negative error."[1] While Leonardo uses "minus" also to indicate an operation (of subtraction), he does not so use the word "plus." Thus, 7 + 4 is written "septem et quatuor." The word "plus," signifying the operation of addition, was first found by Eneström in an Italian algebra of the fourteenth century. The words "plus" and "minus," or their equivalents in the modern tongues, were used by Pacioli, Chuquet, and Widmann. It is proved that the sign + comes from the Latin *et*, as it was cursively written at the time of the invention of printing. The origin of the sign − is still uncertain. These signs were used in Italy by Leonardo da Vinci very soon after their appearance in Widmann's work. They were employed by Grammateus (Heinrich Schreiber), a teacher at the University of Vienna, by Christoff Rudolff in his algebra, 1525, and by Stifel in 1553. Thus, by slow degrees, their adoption became universal.

During the early half of the sixteenth century some of the most prominent German mathematicians (Grammateus, Rudolff, Apian, Stifel) contributed toward the preparation of

[1] G. Eneström in *L'Intermédiaire des Mathématiciens*, 1894, pp. 119–120. Regarding the supposed origin of + and − consult also Eneström in *Öfversigt af Kongl. Vetenskaps – Akademiens Förhandlingar*, Stockholm, 1894, pp. 243–256; Cantor, II., 211–212; De Morgan in *Philosophical Magazine*, 20, 1842, pp. 135–137; in *Trans. of the Philos. Soc. of Cambridge*, 11, p. 203; Eneström, *Biblioth. Math.* (3) 9, p. 155; 10, p. 182.

practical arithmetics, but after that period this important work fell into the hands of the practitioners alone.[1] The most popular of the early text-book writers was *Adam Riese*, who published several arithmetics, but that of 1522 is the one usually associated with his name.

A French work which in point of merit ranks with Pacioli's *Summa de Arithmetica*, but which was never printed before the nineteenth century, is *Le Triparty en la science des nombres*, written in 1484 in Lyons by *Nicolas Chuquet*.[2] A contemporary of Chuquet, in France, was *Jacques Lefèvre*, who brought out printed editions of older mathematical works. For instance, in 1496 there appeared in print the arithmetic of the German monk, *Jordanus Nemorarius*, a work modelled after the arithmetic of Boethius, and at this time over two centuries old. A quarter of a century later, in 1520, appeared a popular French arithmetic by *Estienne de la Roche*, named also *Villefranche*. The author draws his material mainly from Chuquet and Pacioli.[3]

We proceed now to the discussion of a few arithmetical topics. Down to the seventeenth century great diversity and clumsiness prevailed in the numeration of large numbers. Italian authors grouped digits into periods of six, others sometimes into periods of three. Adam Riese, who did more than any one else in the first half of the sixteenth century toward spreading a knowledge of arithmetic in Germany, writes $8\dddot{6}$ $78\dot{9}$ $32\dot{5}$ 178, and reads, "Sechs und achtzig tausend, tausend mal tausend, sieben hundert tausend mal tausendt, neun vnnd achtzig tausend mal tausend, drei hunJert tausent, fünff vnnd zwantzig tausend, ein hundert, acht

[1] Unger, p. 44.

[2] The *Triparty* is printed in *Bulletin Boncompagni*, XIII., 585–592. A description of the work is given by Cantor, II., 347–355.

[3] Cantor, II., 371.

und siebentzig."[1] Stifel in 1544 writes $2^{\cdot}\ 329^{\cdot}\ 089^{\cdot}\ 562^{\cdot}\ 800$, and reads, "duo millia millies millies millies; trecenta viginti novem millia millies millies; octoginta novem millia millies; quingenta sexaginta duo millia; octingenta." Tonstall, in 1522, calls 10^9 "millies millena millia."[2] This habit of grouping digits, for purposes of numeration, did not exist among the Hindus. They had a distinct name for each successive step in the scale, and it has been remarked that this fact probably helped to suggest to them the principle of local value. They read 86789325178 as follows: "8 kharva, 6 padma, 7 vyarbuda, 8 kôti, 9 prayuta, 3 laksha, 2 ayuta, 5 sahasra, 1 çata, 7 daçon, 8."[3] One great objection to this Hindu scheme is that it burdens the memory with too many names.

The first improvement on ancient and mediæval methods of numeration was the invention of the word *millione* by the Italians in the fourteenth century, to signify *great thousand*,[4] or 1000^2. This new word seems originally to have indicated a concrete measure, 10 barrels of gold.[5] The words *millione*, *nulla* or *cero* (zero) occur for the first time in print in the *Arithmetica* of Borgi (1484). In the next two centuries the use of *millione* spread to other European countries. Tonstall, in 1522, speaks of the term as common in England, but rejects it as barbarous! The seventh place in numeration he calls "millena millia; vulgus millionem barbare vocat."[7] Dücange of Rymer mentions the word *million* in 1514;[8] in 1540 it occurs once in the arithmetic of Christoff Rudolff.

The next decided advance was the introduction of the words

[1] Wildermuth, article "Rechnen" in *Encyklopædie des gesammten Erziehungs- und Unterrichtswesens*, Dr. K. A. Schmid, 1885, p. 794.
[2] Peacock, p. 426.
[3] Hankel, p. 15.
[4] Peacock, p. 378.
[5] Hankel, p. 14.
[6] Cantor, II., 305.
[7] Peacock, p. 426.
[8] Wildermuth.

billion, trillion, etc. Their origin dates back almost to the time when the word *million* was first used. So far as known, they first occur in a manuscript work on arithmetic by that gifted French physician of Lyons, *Nicolas Chuquet.* He employs the words *byllion, tryllion, quadrillion, quyllion, sixlion, septyllion, octyllion, nonyllion,* "et ainsi des aultres se plus oultre on voulait proceder," to denote the second, third, etc. powers of a *million, i.e.* $(1000{,}000)^2$, $(1000{,}000)^3$, etc.[1] Evidently Chuquet had solved the difficult question of numeration. The new words used by him appear in 1520 in the printed work of *La Roche.* Thus the great honour of having simplified numeration of large numbers appears to belong to the French. In England and Germany the new nomenclature was not introduced until about a century and a half later. In England the words *billion, trillion,* etc., were new when Locke wrote, about 1687.[2] In Germany these new terms appear for the first time in 1681 in a work by *Heckenberg* of Hanover, but they did not come into general use before the eighteenth century.[3]

About the middle of the seventeenth century it became the custom in France to divide numbers into periods of three digits, instead of six, and to assign to the word *billion,* in place of the old meaning, $(1000{,}000)^2$ or 10^{12}, the new meaning of 10^9.[4] The words *trillion, quadrillion,* etc., receive the new definitions of 10^{12}, 10^{15}, etc. At the present time the words *billion, trillion,* etc., mean in France, in other south-European countries, and in the United States (since the first quar-

[1] Cantor, II., 348.

[2] Locke, *Human Understanding,* Chap. XVI.

[3] Unger, p. 71.

[4] *Dictionnaire de la Langue Française par* E. Littré. It is interesting to notice that Bishop Berkeley, when a youth of twenty-three, published in Latin an arithmetic (1707), giving the words *billion, trillion,* etc., with their new meanings.

ter of this century), 10^9, 10^{12}, etc.; while in Germany, England, and other north-European countries they mean 10^{12}, 10^{18}, etc.

From what has been said it appears that, while the Arabic notation of integral numbers was brought to perfection by the Hindus as early as the ninth century, our present numeration dates from the close of the fifteenth century. One of the advantages of the Arabic notation is its independence of its numeration. At present the numeration, though practically adequate, is not fully developed. To read a number of, say, 1000 digits, or to read the value of π, calculated to 707 decimal places by William Shanks, we should have to invent new words.

A good numeration, accompanied by a good notation, is essential for proficient work in numbers. We are told that the Yancos on the Amazon could not get beyond the number three, because they could not express that idea by any phraseology more simple than Poettarrarorincoaroac.[1]

As regards arithmetical operations, it is of interest to notice that the Hindu custom was introduced into Europe, of beginning an addition or subtraction, sometimes from the right, but more commonly from the left. Notwithstanding the inconvenience of this latter procedure, it is found in Europe as late as the end of the sixteenth century.[2]

Like the Hindus, the Italians used many different methods in the multiplication of numbers. An extraordinary passion seems to have existed among Italian practitioners of arithmetic, at this time, for inventing new forms. Pacioli and Tartaglia speak slightingly of these efforts.[3] Pacioli himself gives eight methods and illustrates the first, named *bericuocoli* or *scacherii*, by the first example given here:[4]

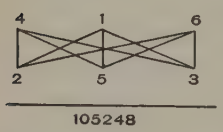
[1] Peacock, p. 390.
[2] Peacock, p. 427.
[3] Peacock, p. 431.
[4] Peacock, p. 429; Cantor, II., 311.

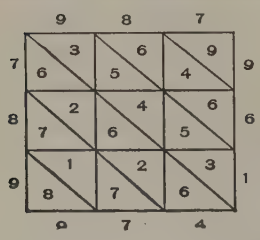
9876	9876
6789	6789
8 8 8 8 4	61101000
7 9 0 0 8	5431200
6 9 1 3 2	475230
5 9 2 5 6	40734
67048164	67048164

The second method, for some unknown reason called *castellucio, i.e.* "by the little castle," is shown here in the second example. The third method (not illustrated here) invokes the aid of tables; the fourth, *crocetta sine casella, i.e.* "by cross multiplication," though harder than the others, was practised by the Hindus (named by them the "lightning" method), and greatly admired by Pacioli. See our third example, in which the product is built up as follows:

$$3 \cdot 6 + 10(3 \cdot 1 + 5 \cdot 6) + 100(3 \cdot 4 + 5 \cdot 1 + 2 \cdot 6) + 1000(5 \cdot 4 + 2 \cdot 1) + 10{,}000(2 \cdot 4).$$

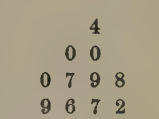
In Pacioli's fifth method, *quadrilatero,* or "by the square," the digits are entered in a square divided up like a chessboard in Hindu fashion, and are added diagonally. His sixth is called *gelosia* or *graticola, i.e.* "latticed multiplication" (see our fourth example, 987×987). It is so named because the figure looks like a lattice or grating, such as was then placed in Venetian windows that ladies and nuns might not easily be seen from the street.[1] The word *gelosia* means primarily jealousy. The last two of Pacioli's methods are illustrated, respectively, by the examples $234 \times 48 = 234 \times 6 \times 8$ and $163 \times 17 = 163 \times 10 + 163 \times 7$.





[1] Peacock, p. 431.

Some books also give the mode of multiplication devised by the early Arabs in imitation of one of the Hindu methods. We illustrate it in the example[1] $359 \times 837 = 300483$. It would be erroneous to conclude from the existence of these and other methods that all of them were in actual use. As a matter of fact the first of Pacioli's methods (now in common use) was the one then practised almost exclusively. The second method of Pacioli, illustrated in our second example, would have been a better choice, as we expect to show later.



It is worthy of remark that the Hindus and Arabs apparently did not possess a multiplication *table*, such, for instance, as the one given by Boethius,[2] which was arranged in a square. We show it here as far as 4×4. The Italians gave this table in their arithmetics. Another form of it, the triangular, shown here as far as 4×4, sometimes occurs in arithmetical books. Some writers (for instance, Finæus in France and Recorde in England) teach also a kind of complementary multiplication, resembling a process first found among the Romans. It is frequently called the "sluggard's rule," and was intended to relieve the memory of all products of digits exceeding 5. It is analogous to the process in Gerbert's complementary division: "Subtract each digit from 10, and write down the differences together, and add as many tens to their product as the $\left\{\begin{matrix}\text{first}\\\text{second}\end{matrix}\right\}$ digit exceeds the $\left\{\begin{matrix}\text{second}\\\text{first}\end{matrix}\right\}$ difference."[3] If a and b designate the digits, then the rule rests on the identity $(10-a)(10-b)+10(a+b-10)=ab$.

1	2	3	4
2	4	6	8
3	6	9	12
4	8	12	16

			1
		2	4
	3	6	9
4	8	12	16

$$\begin{matrix}7 & 3\\ \times & \\ 6 & 4\end{matrix}$$

[1] Unger, p. 77. [2] Boethius (Friedlein's Ed.), p. 53.
[3] Peacock, p. 432.

Division was always considered an operation of considerable difficulty. Pacioli gives four methods. The first, "division by the head," is used when the divisor consists of one digit or two digits (such as 12, 13), included in the Italian tables of multiplication.[1] In the second, division is performed successively by the simple factors of the divisor. The third method, "by giving," is so called because after each subtraction we *give* or *add* one more figure on the right hand. This is the method of "long division" now prevalent. But Pacioli expended his enthusiasm on the fourth method, called by the Italians the "galley," because the digits in the completed work were arranged in the form of that vessel. He considered this procedure the swiftest, just as the galley was the swiftest ship. The English call it the *scratch* method. The complete division of 59078 by 74 is shown in Fig. 1.[2]

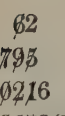


Fig. 1.

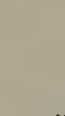


Fig. 2.

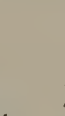


Fig. 3.

Fig. 4.

Fig. 5.

The other four figures show the division in its successive stages.[3]

[1] Peacock, p. 432.

[2] This example is taken from the early German mathematician, Purbach, by Arno Sadowski, *Die österreichiche Rechenmethode*, Königsberg, 1892, p. 14. Pacioli's illustration of the *galley* method is given by Peacock, p. 433, Unger, p. 79.

[3] The work proceeds as follows: Fig. 2 shows the dividend and divisor written in their proper positions, also a curve to indicate the place for the quotient. Write 7 in the quotient; then $7 \times 7 = 49$, $59 - 49 = 10$; write 10 above, scratch the 59 and also the 7 in the divisor. $7 \times 4 = 28$; the 4

For a long time the *galley* or *scratch* method was used almost to the entire exclusion of the other methods. As late as the seventeenth century it was preferred to the one now in vogue. It was adopted in Spain, Germany, and England. It is found in the works of Tonstall, Recorde, Stifel, Stevin, Wallis, Napier, and Oughtred. Not until the beginning of the eighteenth century was it superseded in England.[1] It will be remembered that the *scratch* method did not spring into existence in the form taught by the writers of the sixteenth century. On the contrary, it is simply the graphical representation of the method employed by the Hindus, who calculated with a coarse pencil on a small dust-covered tablet. The erasing of a figure by the Hindus is here represented by the scratching of a figure. On the Hindu tablet, our example, taken from Purbach, would have appeared, when completed, as follows:

$$\begin{array}{r|l} 26 & \\ 59078 & 798 \\ 74 & \end{array}$$

The practice of European arithmeticians to prove their operations by "casting out the 9's" was another method, useful to the Hindus, but poorly adapted for computation on paper or slate, since in this case the entire operation is exhibited at the close, and all the steps can easily be re-examined.

being under the 0 in the dividend, 28 must be subtracted from 100; the remainder is 72; scratch the 10, the 0 of the dividend, and the 4 of the divisor (Fig. 3); above write the 7 and 2. Write down the divisor one place further to the right, as in Fig. 4. Now 7 into 72 goes 9 times; $9 \times 7 = 63$; $72 - 63 = 9$; scratch 72 and the 7 below, write 9 above; $9 \times 4 = 36$; $97 - 36 = 61$; scratch the 9 above, write 6 above it; scratch 7 in the dividend and write 1 above it; scratch 7 and 4 below, Fig. 5. Again move the divisor one place to the right. It goes 8 times; $7 \times 8 = 56$; $61 - 56 = 5$; scratch 6 and 1 above and write 5 above 1; $8 \times 4 = 32$; $58 - 32 = 26$; scratch the 5 above and write 2; above the scratched 8 in the dividend write 6, Fig. 1. The remainder is 26.

[1] Peacock, p. 434.

As there were two rival methods of division ("by giving" and "galley"), there were also two varieties in the extraction of square and cube root.[1] In case of surds, great interest was taken in the discovery of rules of approximation. Leonardo of Pisa, Tartaglia, and others give the Arabic rule (found, for instance, in the works of the Arabs, Ibn Albannâ and Alḳalṣâdi), which may be expressed in our algebraic symbols thus:[2]

$$\sqrt{a^2 - x} = a + \frac{x}{2a}.$$

This yields the root in excess, while the following Arabic rule makes it too small:

$$\sqrt{a^2 - x} = a + \frac{x}{2a + 1}.$$

Similar formulæ were devised for cube root.

In other methods of approximation to the roots of surds, the idea of decimal fractions makes its first appearance, though their true nature and importance were overlooked. About the middle of the twelfth century, John of Seville, presumably in imitation of Hindu methods, adds $2n$ ciphers to the number, then finds the square root, and takes this as a numerator of a fraction whose denominator is 1, followed by n ciphers. The same method was followed by Cardan, but it failed to be generally adopted, even by his Italian contemporaries; for otherwise it certainly would have been at least mentioned by Cataldi (died 1626) in a work devoted exclusively to the extraction of roots. Cataldi finds the square root by means of continued fractions — a method ingenious and novel, but for practical purposes inferior to Cardan's. Orontius Finæus, in France, and William Buckley (died about 1550), in England,

[1] For an example of square root by the *scratch* method, see Peacock, p. 436.

[2] Consult Cantor, I., 808; Peacock, p. 436.

extracted the square root in the same way as Cardan. In finding the square root of 10, Finæus adds six ciphers, and concludes as shown here. The 3 | 162 expresses the square root in decimals. A new branch of arithmetic — decimal fractions — thus stared him in the face, as it had many of his predecessors and contemporaries! But he sees it not; he is thinking of sexagesimal fractions, and hastens to reduce the fractional part to sexagesimal divisions of an integer,[1] thus, $3 \cdot 9' \cdot 43'' \cdot 12'''$. What was needed for the discovery of decimals in a case like this? *Observation, keen observation.* And yet certain philosophers would make us believe that observation is not needed or developed in mathematical study!

```
10000000( 3 | 162
          | 60
        9 | 720
          | 60
       43 | 200
          | 60
       12 | 000
```

Close approaches to the discovery of decimals were made in other ways. The German Christoff Rudolff performed divisions by 10, 100, 1000, etc., by cutting off by a comma ("mit einer virgel")[2] as many digits as there are zeros in the divisor.

The honour of the invention of decimal fractions belongs to *Simon Stevin* of Bruges in Belgium (1548–1620), a man remarkable for his varied attainments in science, for his independence of thought, and extreme lack of respect for authority. It would be interesting to know exactly how he came upon his great discovery. In 1584 he published in Flemish (later in French) an interest table. "I hold it now next to certain," says De Morgan,[3] "that the same convenience which has always dictated the decimal form for tables of compound interest was the origin of decimal fractions themselves." In 1585 Stevin published his *La Disme* (the fourth part of a

[1] Peacock, p. 437. [2] Cantor, II., 399.
[3] *Arithmetical Books*, p. 27.

French work on mathematics), covering only seven pages, in which decimal fractions are explained. He recognized the full importance of decimal fractions, and applied them to all the operations of ordinary arithmetic. No invention is perfect at its birth. Stevin's decimal fractions lacked a suitable notation. In place of our decimal point, he used a cipher; to each place in the fraction was attached the corresponding index. Thus, in his notation, the number 5.912 would be $5^{0}9^{1}1^{2}2^{3}$ or $5\textcircled{0}9\textcircled{1}1\textcircled{2}2\textcircled{3}$. These indices, though cumbrous, are interesting because herein we shall find the principle of another important innovation made by Stevin — an exponential notation. As an illustration of Stevin's notation, we append the following division.[1]

(0) (1) (2) (3) (4) (5) (1) (2)
3 4 4 3 5 2 by 9 6

1
1 8 8
5 1 1 4
7 8 3 7
3 4 4 3 5 2 ((0) (1) (2) (3) 3 5 8 7
9 8 8 8
9 9 9

He was enthusiastic, not only over decimal fractions, but also over the decimal division of weights and measures. He considered it the duty of governments to establish the latter. As to decimals, he says, that, while their introduction may be delayed, "it is certain that if the nature of man in the future remains the same as it is now, then he will not always neglect so great an advantage." His decimals met with ready, though not immediate, recognition. His *La Disme* was translated into English in 1608 by Richard Norton. A decimal arithmetic was published in London, 1619, by Henry Lyte.[2] As to weights and measures, little did Stevin suspect that two hundred years would elapse before the origin of the metric system; and that at the close of the nineteenth century England and the New World would still be hopelessly bound by the chains of custom to the use

[1] Peacock, p. 440. [2] Peacock, p. 440.

of yards, rods, and avoirdupois weights. But we still hope that the words of John Kersey may not be proved prophetic: "It being improbable that such a Reformation will ever be brought to pass, I shall proceed in directing a Course to the Studious for obtaining the frugal Use of such decimal fractions as are in his Powers."[1]

After Stevin, decimals were used on the continent by *Joost Bürgi*, a Swiss by birth, who prepared a manuscript on arithmetic soon after 1592, and by *Johann Hartmann Beyer*, who assumes the invention as his own. In 1603 he published at Frankfurt on the Main a *Logistica Decimalis*. With Bürgi, a zero placed underneath the digit in unit's place answers as a sign of separation. Beyer's notation resembles Stevin's, but it may have been suggested to him by the sexagesimal notation then prevalent. He writes 123.459872 thus:

$$123 \cdot {}^{0}4 \cdot {}^{I}5 \cdot {}^{II}9 \cdot {}^{III}8 \cdot {}^{IV}7 \cdot {}^{V}2.$$

Again he writes .000054 thus, 54^{VI}, and remarks that these differ from other fractions in having the denominator written above the numerator. The decimal point, says Peacock, is due to Napier, who in 1617 published his *Rabdologia* containing a treatise on decimals, wherein the decimal point is used in one or two instances. In the English translation of Napier's *Descriptio*, executed by Edward Wright in 1616, and corrected by the author, the decimal point occurs on the first page of logarithmic tables. There is no mention of decimals in English arithmetics between 1619 and 1631. Oughtred, in 1631, designates .56 thus, $0|\underline{56}$. Albert Girard, a pupil of Stevin, in 1629 uses the point on one occasion. John Wallis, in 1657, writes $12|\underline{345}$, but afterwards in his algebra adopts the

[1] Kersey's Wingate, 16th Ed., London, 1735, p. 119. Wildermuth quotes the same passage from the 2d Ed., 1668.

usual point. Georg Andreas Böckler, in his *Arithmetica nova*, Nürnberg, 1661, uses the comma in place of the point (as do the Germans at the present time), but applies decimals only to the measurement of lengths, surfaces, and solids.[1] De Morgan[2] says that "it was long before the simple decimal point was fully recognized in all its uses, in England at least, and on the continent the writers were rather behind ours in this matter. As long as Oughtred was widely used, that is, till the end of the seventeenth century, there must have been a large school of those who were trained to the notation $123\,|\overline{456}$. To the first quarter of the eighteenth century, then, we must refer, not only the complete and final victory of the decimal point, but also that of the now universal method of performing the operation of division and extraction of the square root." The progress of the decimal notation, and of all other, is interesting and instructive. "The history of language . . . is of the highest order of interest, as well as utility; its suggestions are the best lesson for the future which a reflecting mind can have." (De Morgan.)

To many readers it will doubtless seem that after the Hindu notation was brought to perfection probably in the ninth century, decimal fractions should have arisen at once in the minds of mathematicians, as an obvious extension of it. But[3] "it is curious to think how much science had attempted in physical research, and how deeply numbers had been pondered before it was perceived that the all-powerful simplicity of the 'Arabic Notation' was as valuable and as manageable in an infinitely descending as in an infinitely ascending progression."

The experienced teacher has again and again made observa-

[1] Wildermuth. [2] *Arithmetical Books*, p. 26.

[3] Napier, Mark. *Memoirs of John Napier of Merchiston*, Edinburgh, 1834, Chap. II.

tions similar to this, on the development of thought, in watching the progress of his pupils. The mind of the pupil, like the mind of the investigator, is bent upon the attainment of some fixed end (the solution of a problem), and any consideration not directly involved in this immediate aim frequently escapes his vision. Persons looking for some particular flower often fail to see other flowers, no matter how pretty. One of the objects of a successful mathematical teacher, as of a successful teacher in natural science, should be to habituate students to keep a sharp lookout for other things, besides those primarily sought, and to make these, too, subjects of contemplation. Such a course develops investigators, original workers.

The miraculous powers of modern calculation are due to three inventions: the Hindu Notation, Decimal Fractions, and Logarithms. The invention of logarithms, in the first quarter of the seventeenth century, was admirably timed, for Kepler was then examining planetary orbits, and Galileo had just turned the telescope to the stars. During the latter part of the fifteenth and during the sixteenth century, German mathematicians had constructed trigonometrical tables of great accuracy, but this greater precision enormously increased the work of the calculator. It is no exaggeration to say with Laplace that the invention of logarithms "by shortening the labours doubled the life of the astronomer." Logarithms were invented by *John Napier*, Baron of Merchiston, in Scotland (1550–1617). At the age of thirteen, Napier entered St. Salvator College, St. Andrews. An uncle once wrote to Napier's father, "I pray you, Sir, to send John to the schools either of France or Flanders, for he can learn no good at home." So he was sent abroad. In 1574 a beautiful castle was completed for him on the banks of the Endrick. On the opposite side of the river was a lint mill, and its clack greatly disturbed

Napier. He sometimes desired the miller to stop the mill so that the train of his ideas might not be interrupted.[1] In 1608, at the death of his father, he took possession of Merchiston castle.

Napier was an ardent student of theology and astrology,[2] and delighted to show that the pope was Antichrist. More worthy of his genius were his mathematical studies, which he pursued as pastime for over forty years. Some of his mathematical fragments were published for the first time in 1839. The great object of his mathematical studies was the simplifying and systematizing of arithmetic, algebra, and trigonometry. Students in trigonometry remember "Napier's analogies," and "Napier's rule of circular parts," for the solution of spherical right triangles. This is, perhaps, "the happiest example of artificial memory that is known." In 1617 was published his *Rabdologia*, containing "Napier's rods" or "bones"[3] and other devices designed to simplify multiplication and division. This work was well known on the continent, and for a time attracted even more attention than his logarithms. As late as 1721 E. Hatton, in his arith-

[1] *Dict. Nat. Biog.*

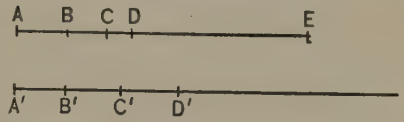
[2] In this connection the title of the following book is interesting. "A Bloody Almanack Foretelling many certaine predictions which shall come to passe this present yeare 1647. With a calculation concerning the time of the day of Judgment, drawne out and published by that famous astrologer, the Lord Napier of Marcheston." For this and for a catalogue of Napier's works, see MACDONALD's Ed. of NAPIER's *Construction of the Wonderful Canon of Logarithms*, 1889.

[3] For a description of Napier's bones, see article "Napier, John," in the *Encyclopædia Britannica*, 9th Ed. In the dedication Napier says, "I have always endeavoured according to my strength and the measure of my ability to do away with the difficulty and tediousness of calculations, the irksomeness of which is wont to deter very many from the study of mathematics." See MACDONALD's Ed. of NAPIER's *Construction*, p. 88.

metic, takes pains to explain multiplication, division, and evolution by "Neper's Bones or Rods."

His logarithms were the result of prolonged, unassisted and isolated speculation. Nowadays we usually say that, in $n = b^x$, x is the logarithm of n to the base b. But in the time of Napier our exponential notation was not yet in vogue. The attempts to introduce exponents, made by Stifel and Stevin, were not yet successful, and Harriot, whose algebra appeared long after Napier's death, knew nothing of indices. It is one of the greatest curiosities of the history of science that Napier constructed logarithms before exponents were used. That logarithms flow naturally from the exponential symbol was not observed until much later by Euler.[1] What, then, was Napier's line of thought?

Let AE be a definite line, $A'D'$ a line extending from A' indefinitely. Imagine two points starting at the same moment; the one moving from A toward E, the other from A' along $A'D'$. Let the velocity during the first moment be the same for both. Let that of the point on line $A'D'$ be uniform; but the velocity of the point on AE decreasing in such a way that when it arrives at any point C, its velocity is proportional to the remaining distance CE. If the first point moves along a distance AC, while the second one moves over a distance $A'C'$, then Napier calls $A'C'$ the logarithm of CE.



This process appears strange to the modern student. Let us develop the theory more fully. Assume a very large initial velocity $= AE =$ (say) v. Then, during every successive short interval or moment of time, measured approximately by $\frac{1}{v}$, the

[1] J. J. Walker, "Influence of Applied on the Progress of Pure Mathematics," *Proceedings Lond. Math. Soc.*, XXII., 1890.

lower point will travel approximately unit distance, which is the product of the uniform velocity v and the time $\frac{1}{v}$ (nearly).

The upper point, starting likewise with a velocity $v = AE$, travels during the first moment very nearly unit's distance AB, and arrives at B with a velocity $= BE = v - 1 = v\left(1 - \frac{1}{v}\right)$. During the second moment of time the velocity of the upper point is very nearly $v - 1$, hence the distance BC is $\frac{v-1}{v}$, and the distance $CE = BE - BC = v - 1 - \frac{v-1}{v} = v\left(1 - \frac{1}{v}\right)^2$. The distance of the point from E at the end of the third moment is similarly found to be $v\left(1 - \frac{1}{v}\right)^3$, and after the v^{th} moment, $v\left(1 - \frac{1}{v}\right)^v$. The distances from E of the upper point at the end of successive moments are, therefore, represented by the first of the two following series (approximately),

$$v,\ v\left(1-\frac{1}{v}\right),\ v\left(1-\frac{1}{v}\right)^2,\ v\left(1-\frac{1}{v}\right)^3,\ \cdots,\ v\left(1-\frac{1}{v}\right)^v,$$

$$0,\qquad 1,\qquad 2,\qquad 3,\qquad \cdots,\qquad v.$$

The second series represents at the end of corresponding intervals of time very nearly the distances of D' from A'. According to Napier, the numbers in the lower series are approximately the logarithms of the corresponding numbers in the upper series. Now observe that the lower series is an arithmetical progression and the upper a geometrical progression. It is here that Napier's discovery comes in touch with the work of previous investigators, like Archimedes and Stifel; it is here that the continuity between the old and the new exists.

The relation between numbers and their logarithms, which

is indicated by the above series, is found, of course, in the logarithms now in general use. The numbers in the geometric series, 1, 10, 100, 1000, have for their common logarithms (to the base 10), the numbers in the arithmetic series, 0, 1, 2, 3. But observe one very remarkable peculiarity of Napier's logarithms: they increase as the numbers themselves *decrease* and numbers exceeding v have *negative* logarithms. Moreover, zero is the logarithm, not of unity (as in modern logarithms), but of v, which was taken by Napier equal to 10^7. Napier calculated the logarithms, not of successive integral numbers, from 1 upwards, but of *sines*. His aim was to simplify trigonometric computations. The line AE was the sine of 90° (*i.e.* of the radius) and was taken equal to 10^7 units. BE, CE, DE, were sines of arcs, and $A'B'$, $A'C'$, $A'D'$ their respective logarithms. It is evident from what has been said that the logarithms of Napier are not the same as the natural logarithms to the base $e = 2.718\cdots$. This difference must be emphasized, because it is not uncommon for text-books on algebra to state that the natural logarithms were invented by Napier.[1] The relation existing between natural logarithms and those of Napier is expressed by the formula,[2]

$$\text{Nap. log } y = 10^7 \text{ nat. log } \frac{10^7}{y}.$$

It must be mentioned that Napier did not determine the

[1] In view of the fact that German writers of the close of the last century were the first to point out this difference, it is curious to find in *Brockhaus' Konversations Lexikon* (1894), article "Logarithmus," the statement that Napier invented natural logarithms. For references to articles by early writers pointing out this error, consult Dr. S. Günther, *Vermischte Untersuchungen*, Chap. V., or my *Teaching and History of Mathematics in the United States*, p. 390.

[2] For its derivation see Cajori, *History of Mathematics*, p. 163.

base to his system of logarithms. The notion of a "base" not only never suggested itself to him, but in fact is inapplicable to his system, unless it is modified somewhat.[1]

Napier's great invention was given to the world in 1614, in a work entitled, *Mirifici logarithmorum canonis descriptio.*[2] In it he explained the nature of logarithms, and gave a logarithmic table of the natural sines of a quadrant from minute to minute. In 1619 appeared Napier's *Mirifici logarithmorum canonis constructio*, as a posthumous work,[3] in which his method of calculating logarithms is explained.[4] The follow-

[1] That the notion of a "base" may become applicable, it is necessary that zero be the logarithm of 1 and not of 10^7. This relation $\log 1 = 0$ is obtained approximately if each term of Napier's geometric series and each term of his arithmetic series is divided by v or 10^7. This gives us

$$1,\ \left(1-\frac{1}{10^7}\right),\ \left(1-\frac{1}{10^7}\right)^2,\ \left(1-\frac{1}{10^7}\right)^3,\ \ldots,\ \left(1-\frac{1}{10^7}\right)^{10^7},$$

$$0,\qquad \frac{1}{10^7},\qquad \frac{2}{10^7},\qquad \frac{3}{10^7},\quad \ldots,\qquad 1.$$

Here 1 appears as the logarithm of $\left(1-\frac{1}{10^7}\right)^{10^7}$, which is nearly equal to e^{-1}, where $e = 2.718\ldots$. Hence the base of Napier's logarithms, *as here modified*, is the reciprocal of the base in the natural system.

[2] From a note at the end of the table of logarithms: "Since the calculation of this table, which ought to have been accomplished by the labour and assistance of many computors, has been completed by the strength and industry of one alone, it will not be surprising if many errors have crept into it." The table is remarkably accurate, as fewer errors have been found than might be expected. See NAPIER'S *Construction* (MACDONALD'S Ed.), pp. 87, 90–96.

[3] It has been republished in Latin at Paris, 1895. An English translation of the *Constructio*, by W. R. Macdonald, appeared in Edinburgh, 1889.

[4] For a *brief* explanation of Napier's mode of computation see CANTOR, II., p. 669.

ing is a copy of part of the first page of the *Descriptio* of 1614:

Gr. 0 + | −
0

Min.	Sinus.	Logarithmi.	Differentiæ.	Logarithmi.	Sinus.	
0	0	Infinitum	Infinitum	0	10000000	60
1	2909	81425681	81425680	1	10000000	59
2	5818	74494213	74494211	2	9999998	58
3	8727	70439564	70439560	4	9999996	57
4	11636	67562746	67562739	7	9999993	56
5	14544	65331315	65331304	11	9999989	55

At the bottom of the first page in the *Descriptio*, on the right, is the figure "89," for 89°. In the columns marked "sinus," we have here copied the natural sines of 0°, 0 to 5 minutes, and of 89°, 55 to 60 minutes. In the columns marked "logarithmi," are the logarithms of these sines, and in the column "differentiæ" the differences between the logarithmic figures in the two columns. Since $\sin x = \cos(90 - x)$, this semi-quadrantal arrangement of the tables really gives all the cosines of angles and their logarithms. Thus, $\log \cos 0° 5' = 11$ and $\log \cos 89° 55' = 65331315$. Moreover, since $\log \tan x = -\log \cot x = \log \sin x - \log \cos x$, the column marked "differentiæ" gives the logarithmic tangents, if taken +, and the logarithmic cotangents if taken −.

Napier's logarithms met with immediate appreciation both in England and on the continent. *Henry Briggs* (1556[1]–1630), who in Napier's time was professor of geometry in Gresham College, London, and afterwards professor at Oxford, was struck with admiration for the book. "Neper, lord of Mark-

[1] The *Dict. of National Biography* gives 1561 as the date of his birth.

inston, hath set my head and hands at work with his new and admirable logarithms. I hope to see him this summer, if it please God, for I never saw a book which pleased me better and made me more wonder." Briggs was an able mathematician, and was one of the few men of that time who did not believe in astrology. While Napier was a great lover of this pseudo-science, "Briggs was the most satirical man against it that hath been known," calling it "a system of groundless conceits." Briggs left his studies in London to do homage to the Scottish philosopher. The scene at their meeting is interesting. Briggs was delayed in his journey, and Napier complained to a common friend, "Ah, John, Mr. Briggs will not come." At that very moment knocks were heard at the gate, and Briggs was brought into the lord's chamber. Almost one-quarter of an hour went by, each beholding the other without speaking a word. At last Briggs began: "My lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy, viz. the logarithms; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy."[1] Briggs suggested to Napier the advantage that would result from retaining zero for the logarithm of the whole sine, but choosing 10^7 for the logarithm of the tenth part of that same sine, *i.e.* of $5° 44' 22''$. Napier said that he had already thought of the change, and he pointed out a slight improvement on Briggs's idea; viz. that zero should be the logarithm of 1, and 10^7 that of the whole sine, thereby making the characteristic of numbers greater than unity positive and not negative, as suggested by Briggs. Briggs admitted this to be more convenient. The invention of "Brig-

[1] Mark Napier's *Memoirs of John Napier*, 1834, p. 409.

gian logarithms" occurred, therefore, to Briggs and Napier independently. The great practical advantage of the new system was that its fundamental progression was accommodated to the base, 10, of our numerical scale. Briggs devoted all his energies to the construction of tables upon the new plan. Napier died in 1617, with the satisfaction of having found in Briggs an able friend to bring to completion his unfinished plans. In 1624 Briggs published his *Arithmetica logarithmica*, containing the logarithms to 14 places of numbers, from 1 to 20,000 and from 90,000 to 100,000. The gap from 20,000 to 90,000 was filled by that illustrious successor of Napier and Briggs, *Adrian Vlacq*, of Gouda in Holland. He published in 1628 a table of logarithms from 1 to 100,000, of which 70,000 were calculated by himself. The first publication of Briggian logarithms of trigonometric functions was made in 1620 by *Edmund Gunter*, a colleague of Briggs, who found the logarithmic sines and tangents for every minute to seven places. Gunter was the inventor of the words *cosine* and *cotangent*. Briggs devoted the last years of his life to calculating more extensive Briggian logarithms of trigonometric functions, but he died in 1630, leaving his work unfinished. It was carried on by the English *Henry Gellibrand*, and then published by Vlacq at his own expense. Briggs divided a degree into 100 parts, but owing to the publication by Vlacq of trigonometrical tables constructed on the old sexagesimal division, the innovation of Briggs remained unrecognized. Briggs and Vlacq published four fundamental works, the results of which "have never been superseded by any subsequent calculations."[1]

[1] For further information regarding logarithmic tables, consult the articles "Tables (mathematical)" in the *Encyclopædia Britannica*, 9th Ed., in the *English Cyclopædia*, in the *Penny Cyclopædia*, and J. W. L. Glaisher in the report of the committee on mathematical tables,

We have pointed out that the logarithms published by Napier are not the same as our natural logarithms. The first appearance of natural logarithms (with the decimal point omitted) is in an anonymous *Appendix* printed in the 1618 edition of Edward Wright's translation of Napier's *Descriptio.* This interesting *Appendix* explains methods of interpolation and is very probably written by William Oughtred. These interpolations were effected with the aid of a small table containing the logarithms of 72 sines. In it, log 10 is given as 230 2584. In modern notation, $\log_e 10 = 2.302584$.

It is frequently stated that natural logarithms appear in the *New Logarithmes*, published at London by a London teacher of mathematics, *John Speidell*, in the year 1619. The book contains logarithms of sines, tangents, and secants. These are not quite the natural logarithms, on account of complications arising from the fact that the logarithms appear in the tables as integral numbers and also that the natural functions (not printed by Speidell) were taken as integral numbers. Thus, Napier gives *sin* 30′ = 87265, the radius being 10^7. In reality, *sin* 30′ = .0087265. The natural logarithm of this fraction is approximately $\bar{5}.25861$. Adding 10 gives 5.25861. Speidell writes in his tables, *log sin* 30′ = 525 861. The relation between the natural logarithms and the logarithms in Speidell's trigonometric tables is shown by the formula, Sp. log $x = 10^5\left(10 + \log_e \frac{x}{10^5}\right)$. For secants and the latter half of the tangents the addition of 10 is omitted. In Speidell's table *log tan* 89° = 404 812, the natural logarithm of *tan* 89° being 4.04812.[1]

published in the *Report of the British Association for the Advancement of Science for* 1873, pp. 1–175.

[1] On Speidell, see *Quarterly Journal of Pure and Appl. Math.*, Vol. 46, 1915, pp. 174–178; article "Tables" in the *English Cyclopædia;* Report on Tables in the *Report of the British Association for* 1873.

From Speidell's statements it is evident that no theoretical, but purely practical reasons induced him to modify Napier's tables. He desired to simplify logarithms, so that persons ignorant of the algebraic rules of addition and subtraction could use the tables. His tables of 1619 avoid negative numbers, but the important adaptation to the Arabic Notation, brought about in the Briggian system, escaped him. His son, Euclid Speidell, says that he "at last concluded that the decimal or Briggs' logarithms were the best sort for a standard logarithm." Editions of Speidell's book appear to have been issued in 1620, 1621, 1623, 1624, 1627, 1628, but not all by himself. In his "Briefe Treatise of Sphæricall Triangles" he mentions and complains of those who had printed his work without an atom of alteration and yet dispraised it in their prefaces for want of alterations. To them he says:

> "If that thou canst amend it
> So shall the arte increase:
> If thou canst not: commend it,
> Else, preethee hould thy peace."

This unfair treatment of himself Speidell attributes to his not having been at Oxford or Cambridge — "not hauing seene one of the Vniuersities."

In the edition of 1622 of his *New Logarithmes*, Speidell published also logarithms of the numbers 1–1000; these are genuinely natural logarithms, except for the omission of the decimal point. We have mentioned the very small table of 1618, which contains 72 logarithms. Speidell's is the earliest more extensive printed table of natural numbers. There is a column in the table which shows that he means to use his table in calculations by feet, inches, and quarters. Thus the number 775 has 16·1·3 opposite to it, there being 775 quarter inches in 16 feet 1 inch, 3 quarters. This is an interesting example of the efforts made from time to time to partially overcome the

inconvenience arising from the use of different scales in the system of measures from that of our notation of numbers. At the bottom of each page Speidell inserts the logarithms of 100 and 1000 for use in decimal fractions.

The most elaborate system of natural logarithmic tables is Wolfram's, which practically gives natural logarithms of all numbers from 1 to 10,000 to 48 decimal places. They were published in 1778 in J. C. Schulze's *Sammlung.*[1] Wolfram was a Dutch lieutenant of artillery and his table represents six years of very toilsome work. The most complete table of natural logarithms, as regards range, was published by the German lightning calculator Zacharias Dase, at Vienna, in 1850. A table is found also in Rees's Cyclopædia (1819), article, "Hyperbolic Logarithms."

In view of the fact that all early efforts were bent, not toward devising logarithms beautiful and simple in theory, but logarithms as useful as possible in computation, it is curious to see that the earliest systems are comparatively weak in practical adaptation, although of great theoretical interest. Napier almost hit upon natural logarithms, whose modulus is unity, while Speidell luckily found the nugget.

The only possible rival of John Napier in the invention of logarithms was the Swiss Joost Bürgi or Justus Byrgius (1552–1632). In his youth a watchmaker, he afterwards was at the observatory in Kassel, and at Prague with Kepler. He was a mathematician of power, but could not bring himself to publish his researches. Kepler attributes to him the discovery of decimal fractions and of logarithms. Bürgi published a crude table of logarithms six years after the appearance of Napier's *Descriptio*, but it seems that he conceived the idea and constructed that table as early, if not earlier, than Napier

[1] Article "Tables" in the *English Cyclopædia.*

did his.[1] However, he neglected to have the results published until after Napier's logarithms (largely through the influence of Kepler) were known and admired throughout Europe.

The methods of computing logarithms adopted by Napier, Briggs, Kepler, Vlacq, are arithmetical and deducible from the doctrine of ratios. After the large logarithmic tables were once computed, writers like Gregory St. Vincent, Newton, Nicolaus Mercator, discovered that these computations can be performed much more easily with aid of infinite series. In the study of quadratures, Gregory St. Vincent (1584–1667) in 1647 found the grand property of the equilateral hyperbola which connected the hyperbolic space between the asymptotes with the natural logarithms, and led to these logarithms being called "hyperbolic." By this property, Nicolaus Mercator in 1668 arrived at the logarithmic series, and showed how the construction of logarithmic tables could be reduced by series to the quadrature of hyperbolic spaces.[2]

English Weights and Measures

At as late a period as the sixteenth century, the condition of commerce in England was very low. Owing to men's ignorance of trade and the general barbarism of the times, interest on money in the thirteenth century mounted to an enormous rate. Instances occur of 50 % rates.[3] But in course of the next two centuries a reaction followed. Not only were extortions of this sort prohibited by statute, but, in the reign of Henry VII., severe laws were made against taking any interest whatever,

[1] For a description of it see GERHARDT's *Gesch. d. Math. in Deutschland*, 1877, p. 119; see also p. 75.

[2] For various methods of computing logarithms, consult art. "Logarithms" in the *Encyclopædia Britannica*, 9th Ed.

[3] HUME's *History of England*, Chap. XII.

all interest being then denominated *usury*. That commerce did not flourish under these new conditions is not strange. But after the middle of the sixteenth century we find that the invidious word *usury* came to be confined to the taking of exorbitant or illegal interest; 10 % was permitted. What little trade existed up to this time was carried on mainly by merchants of the Hansa Towns — Easterlings as they were called.[1]

But in the latter half of the fifteenth century the use of gunpowder was introduced, the art of printing invented, America discovered. The pulse and pace of the world began to quicken. Even in England the wheels of commerce were gradually set in motion. In the sixteenth century English commerce became brisk. Need was felt of some preparation for business careers. Arithmetic and book-keeping were introduced into Great Britain.

Questions pertaining to money, weights and measures must of necessity have received some attention even in a semi-civilized community. In attempting to sketch their history, we find not only in weights and measures, but also in English coins, traces of Roman influence. The Saxons improved Roman coinage. That of William the Conqueror was apparently on the plan adopted by Charlemagne in the eighth century and is supposed to be derived from the Romans as regards the division of the pound into 20 shillings and the shilling into 12 pence. The same proportions were preserved in the *lira* of Italy, *libra* of Spain, and *livre* of France, all now obsolete. The pound adopted by William the Conqueror was the Saxon *Moneyer's* or *Tower* pound of 5400 grains.[2]

[1] HUME'S *England*, Chap. XXXV.

[2] P. KELLY'S *Universal Cambist*, London, 1835, Vol. I., p. 29. Much of our information regarding English money, weights, and measures is derived from this work.

Notice that the vegetable kingdom supplied the primary unit of mass, the "grain." A pound in weight and a pound in tale (that is, in reckoning) were the same. An amount of silver weighing one pound was taken to be worth one pound in money. This explains the double meaning of the word "pound," first as a weight-unit, secondly as a money-unit. Later the Troy pound (5760 grains) came to be used for weighing precious metals. A Troy pound of silver would therefore contain $21\frac{1}{3}$ of the shillings mentioned above. Between the thirteenth century and the beginning of the sixteenth the shilling was reduced to $\frac{1}{3}$ its ancient weight. A change of this sort would probably have been disastrous, except for the practice which appears to have been observed, of paying by weight and not by tale. If we except one short period, we may say that the new shilling was permitted to vary but little in its fineness. In 1665, in the reign of Charles II., a Troy pound of silver yielded 62 shillings, in 1816 it yielded 66 shillings.[1] A financial expedient sometimes resorted to by governments was the depreciation of the currency, by means of issues of coin which contained much less silver or gold than older coins, but which had *nominally* the same value. This was tried by Henry VIII., who issued money in which the silver was reduced in amount by $\frac{1}{6}$, later $\frac{1}{2}$, and finally $\frac{2}{3}$. In the reign of Edward VI., the amount of silver was reduced by $\frac{3}{4}$, so that a coin contained only $\frac{1}{4}$ the old amount of silver. Seventeen years after the first of these issues Queen Elizabeth called in the base coin and put money of the old character in circulation. Henry VIII.'s experiment was disastrous, and reduced England "from the position of a first rate to that of a third rate power in Europe for more than a century."[2]

[1] KELLY, Vol. I., p. 29.

[2] Article, "Finance," in *Encyclopædia Britannica*, 9th Ed. Consult also FRANCIS A. WALKER, *Money*, Chaps. X. and XI.

Interesting are the etymologies of some of the words used in connection with English money. The name *sterling* seems to have been introduced through the Hansa merchants in London. "In the time of . . . King Richard I., monie coined in the east parts of Germanie began to be of especiall request in England for the puritie thereof, and was called *Easterling* monie, as all the inhabitants of those parts were called *Easterlings*, and shortly after some of that countrie, skillful in mint matters, . . . were sent for into this realme to bring the coine to perfection; which since that time was called of them *sterling*, for *Easterling*" (Camden). In the early coinages the silver penny or sterling was minted with a deep cross. When it was broken into four parts, each was called a *fourth-ing* or *farthing*, the *ing* being a diminutive. Larger silver pieces of four pence were first coined in the reign of Edward III. They were called *greats* or *groats*. In 1663, in the reign of Charles II., new gold coins were issued, $44\frac{1}{2}$ pieces to one Troy pound of the metal. These were called *guineas*, after the new country on the west coast of Africa whence the gold was brought.[1] The guinea varied in its current price from 20 shillings up to 30, until the year 1717, when, on Sir Isaac Newton's recommendation, it was fixed at 21 shillings, its present value.[2]

The history of measures of weight brings out the curious fact that among the Hindus and Egyptians, as well as Italians, English, and other Europeans, the basis for the unit of weight lay usually in the *grain of barley*. This was also a favourite unit of length. The lowest subdivision of the pound, or of

[1] Thomas Dilworth in his *Schoolmaster's Assistant*, 1784 (first edition about 1743) praises English gold as follows, p. 89: "In England, Sums of Mony are paid in the best Specie, viz., Guineas, by which Means 1000 l or more may be put into a small Bag, and conveyed away in the Pocket; but in Sweden they often pay Sums of Mony in Copper, and the Merchant is obliged to send Wheelbarrows instead of Bags to receive it."

[2] Kelly, Vol. I., p. 30.

other similar units, was usually defined as weighing the same as a certain number of grains of barley. That no great degree of accuracy could be secured and maintained on such a basis is evident. The fact that the writings of Greek physicians were widely studied, led to the general adoption in Europe of the Greek subdivisions of the *litra* or pound. The pound contained 12 *ounces;* the lower subdivisions being the *drachm* or *dram* ("a handful"), the *gramma* ("a small weight"), and the *grain.*[1] The Romans translated *gramma* into *scriptulum* or *scrupulum*, which word has come down to us as "scruple." The word *gramme* has been adopted into the metric system. The Greeks had a second pound of 16 ounces, called *mina.* The subdivision of pounds both duodecimally and according to the fourth power of two is therefore of ancient date. During the Middle Ages there was in Europe an almost endless variety of different sizes of the pound, as also of the foot, but the words "pound" and "foot" were adopted by all languages; thus pointing to a common origin of the measures. The various pounds were usually divided into 16 ounces; sometimes into 12. The word "pound" comes from the Latin *pondus.* The word "ounce" is the Latin *uncia*, meaning "a twelfth part." The words "ounce" and "inch" have one common derivation, the former being called *uncia librœ* (*libra*, pound), and the latter *uncia pedis* (*pes*, foot).[2]

On English standards of weights and measures there existed (we are told by an old bishop) good laws before the Conquest, but the laws then, as subsequently, were not well observed. The Saxon Tower pound was retained by William the Conqueror, and served at first both as monetary and as weight unit. There have been repeated alterations in the size of the pounds in use in England. Moreover, several different kinds

[1] Peacock, p. 444. [2] Kelly, Vol. I., p. 20.

were in use for different purposes at one and the same time. The first serious attention to this subject seems to have been given in 1266, statute 51 Henry III., when, "by the consent of the whole realm of England . . . an English penny, called a sterling, round and without any clipping, shall weigh 32 wheat corns in the midst of the ear, and 20 pence do make an ounce, and 12 ounces one pound, and 8 pounds do make a gallon of wine, and 8 gallons of wine do make a London bushel, which is the eighth part of a quarter." According to this statement, the pound consisted of, and was determined by, the weight of 7680 grains of wheat. Moreover, silver coin was apparently taken by weight, and not by tale. No doubt this was a grievance, when the standard of weight was so ill-established. The pound defined above we take to be the same as the Tower pound. In 1527, statute 18 Henry VIII., the Tower pound, which had been used mostly for the precious metals, was abolished, and the Troy established in its place. The earliest statute mentioning the Troy pound is one of 1414, 2 Henry V.; how much earlier it was used is a debated question. The old Tower pound was equivalent to $\frac{5}{6}$ of the Troy. The name *Troy* is generally supposed to be derived from *Troyes* in France, where a celebrated fair was formerly held, and the pound was used. The English Committee (of 1758) on Weights and Measures were of the opinion that Troy came from the monkish name given to London of *Troy-novant*, founded on the legend of Brute.[1] According to this, Troy

[1] According to mythological history, Brute was a descendant of Æneas of ancient Troy, and having inadvertently killed his father, fled to Britain, founded London, and called it *Troy-novant* (New Troy). Spenser writes, "Faery Queen" III., 9,

> For noble Britons sprong from Trojans bold
> And Troy-novant was built of old Troyes ashes cold.

See Brewer's *Dic. of Phrase and Fable.*

weight means "London weight." At about the same period, perhaps, the avoirdupois weight was established for heavy goods. The name is commonly supposed to be derived from the French *avoir-du-pois*, a corrupt spelling introduced for *avoir-de-pois*, and signifying "goods of weight."[1] In the earliest statutes in which the word avoirdupois is used (9 Edward III., in 1335; and 27 Edward III., in 1353), it is applied to the goods themselves, not to a system of weights. The latter statute says, "Forasmuch as we have heard that some merchants purchase avoirdupois woollens and other merchandise by one weight and sell by another, . . . we therefore will and establish that one weight, one measure, and one yard be throughout the land, . . . and that woollens and all manner of avoirdupois be weighed. . . ." In 1532, 24 Henry VIII., it was decreed that "beef, pork, mutton, and veal shall be sold by weight called haverdupois."[2] Here the word designates weight. According to an anonymous arithmetic of 1596, entitled "The Pathway of Knowledge," the "pound haberdepois is parted into 16 ounces; every ounce 8 dragmes, every dragme 3 scruples, every scruple 20 grains." This pound contains the same number (7680) of grains as the statute pound of 1266 and the same subdivisions of the ounce, drachm, and scruple as our present apothecaries' weight. The number of grains in the pennyweight of the old pound was changed from 32 to 24, rendering the number of grains in the pound 5760. Why and when this change was made we do not know. Cocker, Wingate, etc., say in their arithmetics that "32 *grains of wheat* make 24 *artificial grains*."[3] The origin of our apothecaries' weight, it has been suggested,[4]

[1] Murray's *English Dictionary*.

[2] Johnson's *Universal Cyclopædia*, article "Weights and Measures."

[3] Cocker's *Arithmetic*, Dublin, 1714, p. 13; George Shelley's Ed. of Wingate's *Arithmetick*, 16th Ed., 1735, p. 7.

[4] See article "Weights and Measures" in *Penny* or *English Cyclopædia*.

was this: Medicines were dispensed by the old subdivisions of the pound (as given in the "Pathway of Knowledge") and continued to be so after the standard pound (with 24 instead of 32 grains in the pennyweight), which Queen Elizabeth ordered to be deposited in the Exchequer in 1588, supplanted the old pound. There appear, thus, to have been two different pounds avoirdupois, an old and a new. The new of Queen Elizabeth agreed very closely with, and may have originated from, an old merchant's pound.[1] It must be remarked that before the fifteenth century, and even later, the commercial weight in England was the Amsterdam weight, which was then used in other parts of Europe and in the East and West Indies.[2] In Scotland it has been in partial use as late as our century, while in England it held its own until 1815 in fixing assize of bread.[2] This weight tells an eloquent story regarding the extent of the Dutch trade of early times.

Other kinds of pounds are mentioned as in use in Great Britain.[3] Their variety is bewildering and confusing; their histories and relative values are very uncertain. In fact a pound of the same denomination often had different values in different places. The "Pathway of Knowledge," 1596, gives five kinds of pounds as in use: the Tower, the Troy, the "haberdepoys," the subtill, and the foyle. The subtill pound was used by assayers; the foyle was $\frac{4}{5}$ of a Troy and was used for gold foil, wire, and for pearls. In case of gold foil and wire, the workman probably secured his profit by selling $\frac{4}{5}$ of a pound at the price of a pound of bullion. Many varieties of measure arose from the practice of merchants, who, instead of

[1] "Weights and Measures" in *Penny* or in *English Cyclopœdia.*

[2] Kelly, Vol. II., p. 372, note.

[3] Thus Dilworth, in his *Schoolmaster's Assistant*, 1784, p. 38, says: "Raw, Long, Short, China, Morea-Silk, etc., are weighed by a great Pound of 24 oz. But Ferret, Filosella, Sleeve-Silk, etc., by the common Pound of 16 oz."

varying the *price* of a given amount of an article, would vary the *amount* of a given name (the pound, say) at a given price.[1]

Ancient measures of length were commonly derived from some part of the human body. This is seen in cubit (length of forearm), foot, digit, palm, span, and fathom. Later their lengths were defined in other ways, as by the width (or length) of barley corns, or by reference to some arbitrarily chosen standard, carefully preserved by the government. The cubit is of much greater antiquity than the foot. It was used by the Egyptians, Assyrians, Babylonians, and Israelites. It was employed in the construction of the pyramid of Gizeh, perhaps 3500 B.C. The foot was used by the Greeks and the Romans. The Roman foot (= 11.65 English inches) was sometimes divided into 12 inches or *unciæ*, but usually into 4 *palms* (breadth of the hand across the middle of the fingers) and each palm into 4 finger-*breadths*. Foot rules found in Roman ruins usually give this digital division.[2] By the Romans, as also by the ancient Egyptians, care was taken to preserve the standards, but during the Middle Ages there arose great diversity in the length of the foot.

As previously stated, the early English, like the Hindus and Hebrews, used the barley corn or the wheat corn, in determining the standards of length and of weight. If it be true that Henry I. ordered the yard to be of the length of his arm, then this is an exception. It has been stated that the Saxon yard was 39.6 inches, and that, in the year 1101, it was shortened by Henry I. to the length of his arm. The earliest English statute pertaining to units of length is that of 1324 (17 Edward II.), when it was ordained that "three barley corns, round and dry, make an inch, 12 inches one foot, three feet a yard." Here the *lengths* of the barley corns are taken, while later, in

[1] "Weights and Measures" in *English Cyclopædia*.

[2] De Morgan, *Arith. Books*, p. 5.

the sixteenth century, European writers take the *breadth;* thus, 64 breadths to one "geometrical" foot (Clavius). The breadth was doubtless more definite than the length, especially as the word "round" in the law of 1324 leaves a doubt as to how much of the point of the grain should be removed before it can be so called.

It would seem as though uniformity in the standards should be desired by all honest men, and yet the British government has always experienced great difficulty in enforcing it. To be sure, the provisions of law often increased the confusion. Thus, in 1437, by statute 15 Henry VI., the alnager, or measurer by the ell, is directed "to procure for his own use a cord twelve yards twelve inches long, adding a quarter of an inch to each quarter of a yard." This law marks the era when the woollen manufacture became important, and the law was intended to make certain the hitherto vague custom of allowing the width of the thumb on every yard, for shrinkage. In 1487, as if to repeal this, it was ordained that "cloths shall be wetted before they are measured, and not again stretched." But in a later year the older statute is again followed.[1]

There was no defined relation between the measures of length and of capacity, until 1701. The statute for that year declares that "the Winchester bushel shall be round with a plane bottom, 18½ inches wide throughout, and 8 inches deep." In measures of capacity there was greater diversity than in the units of length and weight, notwithstanding the fact that in the laws of King Edgar, nearly a century before the Conquest, an injunction was issued that one measure, the Winchester, should be observed throughout the realm. Heavy fines were imposed later for using any except the standard bushel, but without avail. The history[2] of the wine gallon illustrates

[1] *North American Review*, No. XCVII., October, 1837.
[2] *North American Review*, No. XCVII.

how standards are in danger of deterioration. By a statute of Henry III. there was but one legal gallon — the wine gallon. Yet about 1680 it was discovered that for a long time importers of wine paid duties on a gallon of 272 to 282 cu. in., and sold the wine by one of capacity varying from 224 to 231 cu. in.

The British Committee of 1758 on weights and measures seemed to despair of success in securing uniformity, saying "that the repeated endeavours of the legislature, ever since Magna Charta, to compel one weight and one measure throughout the realm never having proved effectual, there seems little to be expected from reviving means which experience has shown to be inadequate."

The latter half of the eighteenth and the first part of the nineteenth century saw the wide distribution in England of standards scientifically defined and accurately constructed. Nevertheless, as late as 1871 it was stated in Parliament that in certain parts of England different articles of merchandise were still sold by different kinds of weights, and that in Shropshire there were actually different weights employed for the same merchandise on different market-days.[1]

The early history of our weights and measures discloses the fact that standards have been chosen, as a rule, by the people themselves, and that governments stepped in at a later period and ordained certain of the measures already in use to be legal, to the exclusion of all others. Measures which grow directly from the practical needs of the people engaged in certain occupations have usually this advantage, that they are of convenient dimensions. The furlong ("furrow-long") is about the average length of a furrow; the gallon and hogshead have dimensions which were well adapted for practical use; shoemakers found the barley corn a not unsuitable subdivision of

[1] Johnson's *Universal Cyclop.*, Art. "Weights and Measures."

the inch in measuring the length of a foot. A very remarkable example of the convenient selection of units is given by De Morgan:[1] That the tasks of those who spin might be calculated more readily, the sack of wool was made 13 tods of 28 pounds each, or 364 pounds. Thus, a pound a day was a tod a month, and a sack a year. But where are the Sundays and holidays? It looks as though the weary "spinster" was obliged to put in extra hours on other days, that she might secure her holiday. Again, "The Boke of Measuryng of Lande," by Sir Richarde de Benese (about 1539), suggests to De Morgan the following passage:[1] "The acre is four roods, each rood is ten *daye-workes*, each daye-worke four perches. So the acre being 40 daye-workes of 4 perches each, and the mark 40 groats of 4 pence each, the aristocracy of money and that of land understood each other easily." In a system like the French, systematically built up according to the decimal scale, simple relations between the units for time, for amount of work done, and for earnings, do not usually exist. This is the only valid objection which can be urged against a system like the metric. On the other hand, the old system needs readjustment of its units every time that some invention brings about a change in the mode of working or a saving of time, and new units must be invented whenever a new trade springs up. Unless this is done, systems on the old plan are, if not ten thousand times, seven thousand six hundred forty-seven times worse than the metric system. Again, the old mode of selecting units leads to endless varieties of units, and to such atrocities as 7.92 inches = 1 link, $5\frac{1}{2}$ yards = 1 rod, $16\frac{1}{2}$ feet = 1 pole, 43560 sq. feet = 1 acre, $1\frac{1}{8}$ hogshead = 1 punch.

The advantages secured by having a uniform scale for weights and measures, which coincides with that of the Arabic

[1] *Arithmetical Books*, p. 18.

Notation, are admitted by all who have given the subject due consideration. Among early English writers whose names are identified with attempted reforms of this sort are Edmund Gunter and Henry Briggs, who, for one year, were colleagues in Gresham College, London. No doubt they sometimes met in conference on this subject. Briggs divided the degree into 100, instead of 60, minutes; Gunter divided the chain into 100 links, and chose it of such length that "the work be more easie in arithmetick; for as 10 to the breadth in chains, so the length in chains to the content in acres." Thus, $\frac{1}{10}$ the product of the length and breadth (each expressed in chains) of a rectangle gives the area in acres.

The crowning achievement of all attempts at reform in weights and measures is the metric system. It had its origin at a time when the French had risen in fearful unanimity, determined to destroy all their old institutions, and upon the ruins of these plant a new order of things. Finally adopted in France in 1799, the metric system has during the present century displaced the old system in nearly every civilized realm, except the English-speaking countries. So easy and superior is the system, that no serious difficulty has been encountered in its introduction, wherever the experiment has been tried. The most pronounced opposition to it was shown by the French themselves. The ease with which the change has been made in other countries, in more recent time, is due, in great measure, to the fact that, before its adoption, the metric system had been taught in many of the schools.

Rise of the Commercial School of Arithmeticians in England

Owing to the backward condition of England, arithmetic was cultivated but little there before the sixteenth century. In the fourteenth century the Hindu numerals began to appear in

Great Britain. A single instance of their use in the thirteenth century is found in a document of 1282, where the word[1] *trium* is written 3*um*. In 1325 there is a warrant from Italian merchants to pay 40 pounds; the body of the document contains Roman numerals, but on the outside is endorsed by one of the Italians, 13XC, that is, 1325. The 5 appears incomplete and inverted, resembling the old 5 in the Bamberg Arithmetic of 1483, and the 5 of the apices of Boethius. The 2 has a slight resemblance to the 2 in the Bamberg Arithmetic. The Hindu numerals of that time were so different from those now in use, and varied so greatly in their form, that persons unacquainted with their history are apt to make mistakes in identifying the digits. A singular practice of high antiquity was the use of the old letter ꝯ for 5. Curious errors sometimes occur in notation. Thus, X2 for 12; XXX1, or 301, for 31. The new numerals are not found in the books printed by Caxton, but in the *Myrrour of the World*, issued by him in 1480, there is a wood-cut of an arithmetician sitting before a table on which are tablets with Hindu numerals upon them.

The use of Hindu numerals in England in the fifteenth century is rather exceptional. Until the middle of the sixteenth century accounts were kept by most merchants in Roman numerals. The new symbols did not find widespread acceptance till the publication of English arithmetics began. As in Italy, so in England, the numerals were used by mercantile houses much earlier than by monasteries and colleges.

The first important arithmetical work of English authorship was published in Latin in 1522 by *Cuthbert Tonstall* (1474–1559). No earlier name is known to us excepting that of John Norfolk, who wrote, about[2] 1340, an inferior treatise on

[1] Our account of the numerals in England is taken from James A. Picton's article *On the Origin and History of the Numerals*, 1874.

[2] W. W. R. Ball, *History of the Study of Mathematics at Cambridge*, Cambridge, 1889, p. 7.

progressions which was printed in 1445, and reissued by Halliwell in his *Rara Mathematica*, London, 1841. Norfolk confounds arithmetical and geometrical progressions, and confines himself to the most elementary considerations.

Tonstall studied at Oxford, Cambridge, and Padua, and drew freely from the works of Pacioli and Regiomontanus. Reprints of his arithmetic, *De arte supputandi*,[1] appeared in England and France, and yet it seems to have been but little known to succeeding English writers.[2] The author states that some years previous he had dealings in money (argentariis) and, not to get cheated, had to study arithmetic. He read everything on the subject in every language that he knew, and spent much time, he says, in licking what he found into shape, *ad ursi exemplum*, as the bear does her cubs. According to De Morgan[2] this book is "decidedly the most classical which ever was written on the subject in Latin, both in purity of style and goodness of matter." The book is a "farewell to the sciences" on the author's appointment to the bishopric of London. A modern critic would say that there is not enough demonstration in this arithmetic, but Tonstall is a very Euclid by the side of most of his contemporaries. Arithmetical results frequently needed, he arranges in tables. Thus, he gives the multiplication table in form of a square, also addition, subtraction, and division tables, and the cubes of the first 10 numbers.

For[3] $\frac{3}{4}$ of $\frac{1}{3}$ of $\frac{1}{2}$ he has the notation $\frac{3}{4}\ \frac{1}{3}\ \frac{1}{2}$. Interesting is his discussion of the multiplication of fractions. We must here premise that Pacioli (like many a school-boy of the present day) was greatly embarrassed[4] by the use of the term

[1] In this book, the pages are not numbered. The earliest known work in which the Hindu numerals are used for numbering the pages is one printed in 1471 at Cologne. See UNGER, p. 16, and KÄSTNER, Vol. I., p. 94.

[2] DE MORGAN, *Arithmetical Books*, p. 13.

[3] CANTOR, II., 476.

[4] PEACOCK, p. 439.

"multiplication" in case of fractions, where the product is less than the multiplicand. That "multiply" means "increase" he proves from the Bible: "Be fruitful and multiply, and replenish the earth" (Gen. i: 28); "I will multiply thy seed as the stars of the heaven" (Gen. xxii: 17). But how is this to be reconciled with the product of fractions? In this way: the unit in the product is of greater *virtue* or *significance;* thus, if $\frac{1}{2}$ and $\frac{1}{2}$ are the sides of a square, then $\frac{1}{4}$ represents the area of the square itself. Later writers encountered the same difficulty, but were not always satisfied with Pacioli's explanation. Tonstall discusses the subject with unusual clearness. He takes $\frac{2}{3} \times \frac{3}{4} = \frac{6}{12}$. "If[1] you ask the reason why this happens thus, it is this, that if the numerators alone are multiplied together the integers appear to be multiplied together, and thus the numerator would be increased too much. Thus, in the example given, when 2 is multiplied into 3, the result is 6, which, if nothing more were done, would seem to be a whole number; however, since it is not the integer 2 that must be multiplied by 3, but $\frac{2}{3}$ of the integer 1 that must be multiplied by $\frac{3}{4}$ of it, the denominators of the parts are in like manner multiplied together; so that, finally, by the division which takes place through multiplication of the denominators (for by so much as the denominator increases, by so much are the parts diminished), the increase of the numerator is corrected by as much as it had been augmented more than was right, and by this means it is reduced to its proper value."

This dispute is instanced by Peacock as a curious example of the embarrassment arising when a term, restricted in meaning, is applied to a general operation, the interpretation of which depends upon the kinds of quantities involved. The

[1] We are translating Tonstall's Latin, quoted by Peacock, p. 439.

difficulty which is encountered in multiplication arises also in the process of division of fractions, for the quotient is larger than the dividend. The explanation of the paradox calls for a clear insight into the nature of a fraction. That, in the historical development, multiplication and division should have been considered primarily in connection with integers, is very natural. The same course must be adopted in teaching the young. First come the easy but restricted meanings of multiplication and division, applicable to whole numbers. In due time the successful teacher causes students to see the necessity of modifying and broadening the meanings assigned to the terms. A similar plan has to be followed in algebra in connection with exponents. First there is given an easy definition applicable only to positive integral exponents. Later, new meanings must be sought for fractional and negative exponents, for the student sees at once the absurdity, if we say that in $x^{\frac{1}{2}}$, x is taken one-half times as a factor. Similar questions repeatedly arise in algebra.

Of course, Tonstall gives the English weights and measures; he also compares English money with the French, etc.

It is worthy of remark that Tonstall, to prevent Tyndale's translation of the New Testament from spreading more widely among the masses, is said to have once purchased and burned all the copies which remained unsold. But the bishop's money enabled Tyndale, the following year, to bring out a second and more correct edition!

A quarter of a century after the first appearance of Tonstall's arithmetic, were issued the writings of *Robert Recorde* (1510–1558). Educated at Oxford and Cambridge, he excelled in mathematics and medicine. He taught arithmetic at Oxford, but found little encouragement, notwithstanding the fact that he was an exceptional teacher of this subject. Migrating to London, he became physician to Edward VI.,

and later to Queen Mary. It does not seem that he met with recompense at all adequate to his merits, for afterwards he was confined in prison for debt, and there died. He wrote several works, of which we shall notice his arithmetic and (elsewhere) his algebra. It is stated that Recorde was the earliest Englishman who accepted and advocated the Copernican theory.[1]

His arithmetic, *The Grounde of Artes*, was published in 1540. Unlike Tonstall's, it is written in English and contains the symbols $+$, $-$, $\angle$. The last symbol he uses to denote equality. In his algebra it is modified into our familiar sign $=$. These three symbols are not used except toward the last, under "the rule of Falsehode." He says[2] "$+$ whyche betokeneth too muche, as this line, $-$, plaine without a crosse line, betokeneth too little." The work is written in the form of a dialogue between master and pupil. Once the pupil says, "And I to youre authoritie my witte doe subdue, whatsoever you say, I take it for true," whereupon the master replies that this is too much, "thoughe I mighte of my Scholler some credence require, yet except I shew reason, I do it not desire." Notice here the rhyme, which sometimes occurs in the book, though the verse is not set off into lines by the printer. In all operations, even those with denominate numbers, he tests the results by "casting out the 9's."[3] The scholar complains that he cannot see the reason for the process. "No more doe you of manye things else," replies the master, arguing that one must first learn the art by concisely worded rules, before the reason can be grasped. This is certainly sound

[1] Ball's *Mathematics at Cambridge*, p. 18.

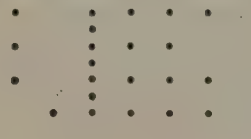
[2] Cantor, II., p. 477–480.

[3] The insufficiency of this test is emphasized by Cocker as follows: "But there may be given a thousand (nay infinite) false products in multiplication, which after this manner may be proved to be true, and therefore this way of proving doth not deserve any example," *Arithmetick*, 28th Ed., 1714, p. 50.

advice. The method of "casting out the 9's" is easily learned, but the reason for it is beyond the power of the young student. It is not contrary to sound pedagogy sometimes to teach merely the facts or rules, and to postpone the reasoning to a later period. Whoever teaches the method and the reasoning in square root *together* is usually less successful than he who teaches the two *apart*, one immediately after the other. It is, we believe, the usual experience, both of individuals and of nations, that the natural order of things is facts first, reasons afterwards. This view is no endorsement of mere memory-culture, which became universal in England after the time of Tonstall and Recorde.

Recorde gives the Rule of Three (or "the golden rule"), progressions, alligation, fellowship, and false position. He thinks it necessary to *repeat* all the rules (like the rule of three, fellowship, etc.) *for fractions.* This practice was prevalent then, and for the succeeding 250 years. He was particularly partial to the rule of false position ("rule of Falsehode") and remarks that he was in the habit of astonishing his friends by proposing hard questions and deducing the correct answer by taking the chance replies of "suche children or ydeotes as happened to be in the place."

It is curious to find in Recorde a treatise on reckoning by counters, "whiche doth not onely serve for them that cannot write and reade, but also for them that can doe both, but have not at some times their pen or tables readie with them." He mentions two ways of representing sums by counters, the *Merchant's* and the *Auditor's* account. In the first, 198 l., 19 s., 11 d., is expressed by counters (our dots) thus: —



$= 100 + 80$ pounds.
$= 10 + 5 + 3$ pounds.
$= 10 + 5 + 4$ shillings.
$= 6 + 5$ pence.

Observe that the four horizontal lines stand respectively for pence, shillings, pounds, and scores of pounds, that counters in the intervening spaces denote half the units in the next line above, and that the detached counters to the left are equivalent to five counters to the right.[1]

The abacus with its counters had ceased to be used in Spain and Italy in the fifteenth century. In France it was used at the time of Recorde, and it did not disappear in England and Germany before the middle of the seventeenth century. The method of abacal computation is found in the English exchequer for the last time in 1676. In the reign of Henry I. the exchequer was distinctly organized as a court of law, but the financial business of the crown was also carried on there. The term "exchequer" is derived from the chequered cloth which covered the table at which the accounts were made up. Suppose the sheriff was summoned to answer for the full annual dues "in money or in tallies." "The liabilities and the actual payments of the sheriff were balanced by means of counters placed upon the squares of the chequered table, those on the one side of the table representing the value of the tallies, warrants, and specie presented by the sheriff, and those on the other the amount for which he was liable," so that it was easy to see whether the sheriff had met his obligations or not. In Tudor times "pen and ink dots" took the place of counters. These dots were used as late as 1676.[2] The "tally" upon which accounts were kept was a peeled wooden rod split in such a way as to divide certain notches previously cut in it. One piece of the tally was given to the payer; the other piece was kept by the exchequer. The transaction could be verified easily by fitting the two halves together and noticing whether

[1] Peacock, p. 410; Peacock also explains the *Auditor's Account*.

[2] Article "Exchequer" in Palgrave's *Dictionary of Political Economy*, London, 1894.

the notches "tallied" or not. Such tallies remained in use as late as 1783.[1]

In the *Winter's Tale* (IV., 3), Shakespeare lets the clown be embarrassed by a problem which he could not do without counters. Iago (in *Othello*, I.) expresses his contempt for Michael Cassio, "forsooth a great mathematician," by calling him a "counter-caster."[2] It thus appears that the old methods of computation were used long after the Hindu numerals were in common and general use. With such dogged persistency does man cling to the old!

While England, during the sixteenth century, produced no mathematicians comparable with Vieta in France, Rheticus in Germany, or Cataldi in Italy, it is nevertheless true that Tonstall and Recorde, through their mathematical works, reflect credit upon England. Their arithmetics are above the average of European works. With the time of Recorde the English began to excel in numerical skill as applied to money. "The questions of the English books," says De Morgan, "are harder, involve more figures in data, and are more skilfully solved." To this fact, no doubt, we must attribute the ready appreciation of decimal fractions, and the instantaneous popularity of logarithms. The number of arithmetical writers in the seventeenth and eighteenth centuries is very large. Among the more prominent of the early writers after Tonstall and Recorde are:[3] William Buckley, mathematical tutor of Edward VI. and author of *Arithmetica Memorativa* (1550); Humfrey Baker, author of *The Well-Spring of the Sciences* (1562); Edmund Wingate, whose *Arithmetick* appeared about 1629; William Oughtred, who in 1631 published his *Clavis*

[1] "Tallies have been at 6, but now at 5 *per cent. per annum*, the interest payable every three months." SHELLEY'S WINGATE'S *Arithmetic*, 1735, p. 407.

[2] PEACOCK, p. 408.

[3] PEACOCK, pp. 437, 441, 442, 452.

Mathematicæ, a systematic text-book on arithmetic and algebra; Noah Bridges, author of *Vulgar Arithmeticke* (1653); Andrew Tacquet, a Jesuit mathematician of Antwerp, author of several books, in particular of *Arithmeticæ Theoria et Praxis* (Antwerp, 1656, later reprinted in London). Mention should be made also of *The Pathway of Knowledge*, an anonymous work, written in Dutch and translated into English in 1596. John Mellis, in 1588, issued the first English work on book-keeping by double entry.[1]

We have seen that the invention of printing revealed in Europe the existence of two schools of arithmeticians, the *algoristic school*, teaching rules of computation and commercial arithmetic, and the *abacistic school*, which gave no rules of calculation, but studied the properties of numbers and ratios. Boethius was their great master, while the former school followed in the footsteps of the Arabs. The algoristic school flourished in Italy (Pacioli, Tartaglia, etc.) and found adherents throughout England and the continent. But it is a remarkable fact that the abacistic school, with its pedantry, though still existing on the continent, received hardly any attention in England. The laborious treatment of arithmetical ratios with its burdensome phraseology was of no practical use to the English merchant. The English mind instinctively rebelled against calling the ratio $3:2=1\frac{1}{2}$ by the name of *proportio superparticularis sesquialtera.*

On the other hand it is a source of regret that the successors of Tonstall and Recorde did not observe the high standard of authorship set by these two pioneers. A decided decline is marked by Buckley's *Arithmetica Memorativa,* a Latin treatise expressing the rules of arithmetic in verse, which, we take it, was intended to be committed to memory. We are glad to be

[1] De Morgan, *Arith. Books*, p. 27.

able to say that this work never became widely popular. The practice of expressing rules in verse was common, before the invention of printing, but the practice of *using them* was not common, else the number of printed arithmetics on this plan would have been much larger than it actually was.[1] Many old arithmetics give occasionally a rhyming rule, but few confine themselves to verse. Few authors are guilty of the folly displayed by Buckley or by Solomon Lowe, who set forth the rules of arithmetic in English hexameter, and in alphabetical order.

In this connection it may be stated that an early specimen of the muse of arithmetic, first found in the *Pathway of Knowledge*, 1596, has come down to the present generation as the most classical verse of its kind:

"Thirtie daies hath September, Aprill, June, and November,
Februarie eight and twentie alone, all the rest thirtie and one."

As a close competitor for popularity is the following stanza[2] quoted by Mr. Davies (Key to Hutton's "Course") from a manuscript of the date 1570 or near it:

"Multiplication is mie vexation
And Division is quite as bad,
The Golden Rule is mie stumbling stule
And Practice drives me mad."

In the sixteenth century instances occur of arithmetics written in the form of questions and answers. During the seventeenth century this practice became quite prevalent both in England and Germany. We are inclined to agree with Wildermuth that it is, on the whole, an improvement on the older practice of simply directing the student to do so and so.

[1] De Morgan, *Arith. Books*, p. 16.
[2] De Morgan, *Arith. Books.*

A question draws the pupil's attention and prepares his mind for the reception of the new information. Unfortunately, the question always relates to *how* a thing is done, never *why* it is done as indicated. It is deplorable to see in the seventeenth century, both in England and Germany, that arithmetic is reduced more and more to a barren collection of rules. The sixteenth century brought forth some arithmetics, by prominent mathematicians, in which attempts were made at demonstration. Then follows a period in which arithmetic was studied solely for commercial purposes, and to this commercial school of arithmeticians (about the middle of the seventeenth century), says De Morgan,[1] "we owe the destruction of demonstrative arithmetic in this country, or rather the prevention of its growth. It never was much the habit of arithmeticians to prove their rules; and the very word *proof*, in that science, never came to mean more than a test of the correctness of a particular operation, by reversing the process, casting out the nines, or the like. As soon as attention was fairly averted to arithmetic for commercial purposes alone, such rational application as had been handed down from the writers of the sixteenth century began to disappear, and was finally extinct in the work of Cocker or Hawkins, as I think I have shown reason for supposing it should be called.[2] From this time began the finished school of teachers, whose pupils ask, when a question is given, what rule it is in, and run away, when they grow up, from any numerical statement, with the declaration that

[1] *Arith. Books*, p. 21.

[2] "Cocker's Arithmetic" was "perused and published" after Cocker's death by John Hawkins. De Morgan claims that the work was not written by Cocker at all, but by John Hawkins, and that Hawkins attached to it Cocker's name to make it sell. After reading the article "Cocker" in the *Dictionary of National Biography*, we are confident in believing Hawkins innocent. Cocker's sudden death at an early age is sufficient to account for most of his works being left for posthumous publication.

anything may be proved by figures — as it may, to them.[1] Anything may be unanswerably propounded, by means of figures, to those who cannot think upon numbers. Towards the end of the last century we see a succession of works, arising one after the other, all complaining of the state into which arithmetic had fallen, all professing to give rational explanation, and hardly one making a single step in advance of its predecessors.

"It may very well be doubted whether the earlier arithmeticians could have given general demonstrations of their processes. It is an unquestionable fact of observation that the application of elementary principles to their apparently most natural deduction, without drawing upon subsequent, or what ought to be subsequent, combinations, seldom takes place at the commencement of any branch of science. It is the work of advanced thought. But the earlier arithmeticians and algebraists had another difficulty to contend with: their fear of their own half-understood conclusions, and the caution with which it obliged them to proceed in extending their half-formed language."

Of arithmetical authors belonging to the commercial school we mention (besides Cocker) James Hodder, Thomas Dilworth, and Daniel Fenning, because we shall find later that their books were used in the American colonies.

James Hodder[2] was in 1661 writing-master in Lothbury, London. He was first known as the author of Hodder's *Arithmetick*, a popular manual upon which Cocker based his better known work. Cocker's chief improvement is the use

[1] The period of teaching arithmetic wholly by rules began in Germany about the middle of the sixteenth century — nearly one hundred years earlier than in England. But the Germans returned to demonstrative arithmetic in the eighteenth century, at an earlier period than did the British. See UNGER, pp. iv, 117, 137.

[2] *Dictionary of National Biography.*

of the new mode of division, "by giving" (as the Italians called it), in place of the "scratch" or "galley" method taught by Hodder. The first edition of Hodder's appeared in 1661, the twentieth edition in 1739. He wrote also *The Penman's Recreation* (the specimens of which are engraved by Cocker, with whom, it appears, Hodder was friendly), and *Decimal Arithmetick*, 1668.

Cocker's *Arithmetick* went through at least 112 editions,[1] including Scotch and Irish editions.[2] Like François Barrême in France and Adam Riese in Germany, Cocker in England enjoyed for nearly a century a proverbial celebrity, these names being synonymous with the science of numbers. A man who influenced mathematical teaching to such an extent deserves at least a brief notice. Edward Cocker (1631–1675) was a practitioner in the arts of writing, arithmetic, and engraving. In 1657 he lived "on the south side of St. Paul's Churchyard" where he taught writing and arithmetic "in an extraordinary manner." In 1664 he advertised that he would open a public school for writing and arithmetic and take in boarders near St. Paul's. Later he settled at Northampton.[3] Aside from

[1] *Dictionary of National Biography.*

[2] "Corrected" editions of Cocker's *Arithmetic* were brought out in 1725, 1731, 1736, 1738, 1745, 1758, 1767 by "George Fisher," a pseudonym for Mrs. Slack. Under the same pseudonym she published in London, 1763, *The Instructor: or Young Man's best Companion, containing spelling, reading, writing, and arithmetic, etc.*, the fourteenth edition of which appeared in 1785 at Worcester, Mass., (the twenty-eighth in 1798) under the title, *The American Instructor:* etc., as above. In Philadelphia the book was printed in 1748 and 1801. Mrs. Slack is the first woman whom we have found engaged in arithmetical authorship. See *Bibliotheca Mathematica*, 1895, p. 75; *Teach. and Hist. of Mathematics in the U. S.*, 1890, p. 12.

[3] Pepys mentions him several times in 1664 in his *Diary*, 10th: "Abroad to find out one to engrave my table upon my new sliding rule with silver plates, it being so small that Browne who made it cannot get one to do it. So I got Cocker, the famous writing master, to do it, and

the absence of all demonstration, Cocker's *Arithmetic* was well written, and it evidently suited the demands of the times. He was a voluminous writer, being the author of 33 works, 23 calligraphic, 6 arithmetical, and 4 miscellaneous. The arithmetical books are: *Tutor to Arithmetick* (1664); *Compleat Arithmetician* (1669); *Arithmetick* (1678; Peacock on page 454 gives the date 1677); *Decimal Arithmetick*, *Artificial Arithmetick* (being of logarithms), *Algebraical Arithmetick* (treating of equations) in three parts, 1684, 1685, "perused and published" by John Hawkins.

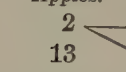
In the English, as well as the French and German arithmetics, which appeared during the sixteenth, seventeenth, and eighteenth centuries, the "rule of three" occupies a central position. Baker says[1] in his *Well-Spring of the Sciences*, 1562, "The rule of three is the chiefest, and the most profitable, and most excellent rule of all arithmeticke. For all other rules have neede of it, and it passeth all other; for the which cause, it is sayde the philosophers did name it the Golden Rule, but now, in these later days, it is called by us the Rule of Three, because it requireth three numbers in the operation." There has been among writers considerable diversity in the notation for this rule. Peacock (p. 452) states the question "If 2 apples cost 3 soldi, what will 13 cost?" and with reference to it, represents Tartaglia's notation thus,

Se pomi 2 ‖ val soldi 3 ‖ che valera pomi 13.

I set an hour by him to see him design it all; and strange it is to see him with his natural eyes to cut so small at his first designing it, and read it all over, without any missing, when for my life I could not, with my best skill, read one word, or letter of it. . . . I find the fellow by his discourse very ingenious; and among other things, a great admirer and well read in the English poets, and undertakes to judge of them all, and that not impertinently." Cocker wrote quaint poems and distichs which show some poetical ability.

[1] Peacock, p. 452.

Recorde and the older English arithmeticians write as follows:



In the seventeenth century the custom was as follows (Wingate, Cocker, etc.):

The notation of this subject received the special attention of Oughtred, who introduced the sign : : and wrote

$$2 \cdot 3 :: 13.$$

M. Cantor[1] says that the dot, used here to express the ratio, later yielded to two dots, for in the eighteenth century the German writer, Christian Wolf, secured the adoption of the dot as the usual symbol of multiplication. In England the reason for the change from dot to colon was a different one. It will be remembered that Oughtred did not use the decimal point. Its general introduction in England took place in the first quarter of the eighteenth century, and we are quite sure that it was the decimal point and not Wolf's multiplication sign which displaced Oughtred's symbol for ratio. As the Germans use a decimal *comma*[2] instead of our point, the reason

[1] Cantor, II., p. 721.

[2] The Germans attribute the introduction of the decimal *comma* to Kepler (see Unger, p. 104; Gerhardt, pp. 78, 109; Günther, *Vermischte Untersuchungen*, p. 133). He uses it in a publication of 1616. Napier, in his *Rabdologia* (1617) speaks of "adding a period or comma," and writes 1993,273 (see *Construction*, Macdonald's Ed., p. 89). English writers did not confine themselves to the decimal *point;* the comma is often used. Thus, Martin's *Decimal Arithmetick* (1763) and Wilder's edition of Newton's *Universal Arithmetick*, London, 1769, use the comma exclusively. In Kersey's Wingate (1735) and in Dilworth (1784),

for the change could not have been the same in Germany as in England. Dilworth[1] does not use the dot in multiplication, but he employs the decimal point and writes[2] proportion once $2 \cdot\cdot 4 :: 8 \cdot\cdot 16$, and on another page $3 : 17 :: 48$. Before the present century, the dot was seldom used by English writers to denote multiplication. If Oughtred had been in the habit of regarding a proportion as the equality of two ratios, then he would probably have chosen the symbol $=$ instead of $::$. The former sign was actually used for this purpose by Leibniz.[3] The notation $2 : 4 = 1 : 2$ was brought into use in the United States and England during the first quarter of the nineteenth century, when Euler's Algebra and French text-books began to be studied by the English-speaking nations.

The rule of three reigned supreme in commercial arithmetics in Germany until the close of the eighteenth century, and in England and America until the close of the first quarter of the present century.[4] It has been much used since. An

we read of the "point or comma," but the point is the sign actually used. In Dodson's Wingate (1760) both the comma and point are used, the latter, perhaps, more frequently. Cocker (1714) and Hatton (1721) do not even mention the comma.

[1] Thomas Dilworth, *Schoolmaster's Assistant*, 22d Ed., London, 1784, page after table of contents; also, pp. 45, 123. The earliest testimonials are dated 1743.

[2] In his time old and new notations were in simultaneous use, for he says, "Some masters, instead of points, use long strokes to keep the terms separate, but it is wrong to do so; for the two points between the first and second terms, and also between the third and fourth terms, shew that the two first and the two last terms are in the same proportion. And whereas four points are put between the second and third terms, they serve to disjoint them, and shew that the second and third, and first and fourth terms are not in the same direct proportion to each other as are these before mentioned." [3] Wildermuth.

[4] Unger (p. 170) says that in Germany the rule of three was preferred as the universal rule for problem-working, during the sixteenth century;

important rôle was played in commercial circles by an allied rule called in English *chain-rule* or *conjoined proportion*, in French *règle conjointe*, and in German *Kettensatz* or *Reesischer Satz*.[1] It received its most perfect formal development and most extended application in Germany and the Netherlands. Kelly[2] attributes the superiority of foreign merchants in the science of exchange to a more intimate knowledge of this rule. The chain-rule was known in its essential feature to the Hindu Brahmagupta; also to the Italians, Leonardo of Pisa, Pacioli, Tartaglia; to the early Germans, Johann Widmann and Adam Riese. In England the rule was elaborated by John Kersey in his edition of 1668 of Wingate's arithmetic. But the one who contributed most toward its diffusion was Kaspar Franz von Rees (born 1690) of Roermonde in Limburg, who migrated to Holland. There he published in Dutch an arithmetic which appeared in French translation in 1737, and in German translation in 1739. In Germany the *Reesischer Satz* became famous. Cocker (28th Ed., Dublin, 1714, p. 232) illustrates the rule by the example: "If 40 l. *Auverdupois* weight at *London* is equal to 36 l. weight at *Amsterdam*, and 90 l. at *Amsterdam* makes 116 l. at *Dantzick*, then how many Pounds at *Dantzick* are equal to 112 l. *Auverdupois* weight at *London?*"

the method, called Practice, being preferred during the seventeenth century, the Chain-Rule during the eighteenth, and Analysis (Bruchsatz, or Schlussrechnung) during the nineteenth. We fail to observe similar changes in England. There the rule of three occupied a commanding position until the present century; the rules of single and double position were used in the time of Recorde more than afterwards; the Chain-Rule never became widely popular; while some attention was always paid to Practice.

[1] The writer remembers being taught the "Kettensatz" in 1873 at the Kantonsschule in Chur, Switzerland, along with three other methods, the "Einheitsmethode" (Schlussrechnung), the "Zerlegungsmethode" (a form of Italian practice) and "Proportion."

[2] Vol. II., p. 3.

(p. 234) "The Terms being disposed according to the 7th Rule foregoing, will stand thus,

	A	B	
l. *at Lond.*	40	36	l. *at Amsterdam*
l. *at Amst.*	90	116	l. *at Dantzick*
		112	l. *at London*

whereby I find that the terms under B multiplied together produce 467712 for a Dividend and the terms under A, *viz.*, 40 and 90 produce 3600 for a Divisor, and Division being finished, the Quotient giveth $129\frac{3}{4}\frac{3}{5}\frac{1}{0}\frac{2}{0}$ Pounds at *Dantzick* for the Answer."

The chain-rule owes its celebrity to the fact that the correct answer could be obtained without any exercise of the mind. Rees reduced the statement to a mere mechanism. While useful to the merchant, the rule was worthless for mind-culture in the school-room. Attempts were made by some arithmetical writers to prove it with aid of proportions or by ordinary analysis. But for pedagogical purposes those attempts proved unsatisfactory. A rule, more apt to lead to errors, but requiring some thought, became known in Germany as "Basedowsche Regel," being recommended by the educational reformer Basedow, though not first given by him. Early in the present century, arithmetical teaching was revolutionized in Germany. The *chain-rule* and *rule of three* were gradually driven into the background, and the "Schlussrechnung" attained more and more prominence, even though Pestalozzi was partial to the use of proportion. The "Schlussrechnung" is sometimes designated in English by the word Analysis. If 3 yards cost \$ 7, what will 19 yards cost? One yard will cost \$ $\frac{7}{3}$, and 19 yards $\frac{\$\ 7 \times 19}{3} = \$\ 44.33$; or by a modification of the above, by "aliquot parts": 18 yards will cost \$ 42; one yard \$ 2.33, and 19 yards, \$ 44.33. This "Schlussrechnung" was known

to Tartaglia, but is, without doubt, very much older; for it is a thoroughly natural method which would suggest itself to any sound and vigorous mind. Its pedagogical value lies in the fact that there is no mechanism about it, that it requires no memorizing of formulæ, but makes arithmetic an exercise in thinking. It is strange indeed that in modern times such a method should have been disregarded by arithmeticians dur ing three long centuries.

English arithmetics embraced all the commercial subjects previously used by the Italians, such as simple and compound interest, the direct rule of three, the inverse rule of three (called by Recorde "backer rule of three"), loss and gain, barter, equation of payments, bills of exchange, alligation, annuities, the rules for single and double position, and the subject of tare, trett, cloff. We let Dilworth (p. 37, 1784) explain the last three terms:

"*Q.* Which are the Allowances usually made in Averdupois great Weight to the Buyer?

A. They are Tare, Trett, and Cloff.

Q. What is Tare?

A. Tare is an Allowance made to the Buyer, for the Weight of the Box, Bag, Vessel, or whatever else contains the Goods bought. . . .

Q. What is Trett?

A. Trett is an Allowance by the Merchant to the Buyer of 4 lb. in 104 lb., that is, the six-and-twentieth Part, for Waste or Dust, in some Sorts of Goods. . . .

Q. What is Cloff?

A. Cloff is an Allowance of 2 lb. Weight to the Citizens of London, on every Draught above 3 C. Weight, on some Sorts of Goods, as Galls, Madder, Sumac, Argol, etc.[1]

Q. What are these Allowances called beyond the Seas?

A. They are called the *Courtesies of London;* because they are not practiced in any other Place."

[1] The term "cloff" had also a more general meaning, denoting a small allowance made on goods sold in gross, to make up for deficiences in weight when sold in retail. PEACOCK, p. 455.

To the above subjects, which were borrowed from the Italians, English writers added their own weights and measures, and those of the countries with which England traded. In the seventeenth century were added decimal fractions, which were taught in books more assiduously then than a century later. It is surprising to find that some arithmetics devoted considerable attention to logarithms. Cocker wrote a book on "Artificial Arithmetick." We have seen the rules for the use of logarithms, together with logarithmic tables of numbers, in the arithmetics of John Hill (10th Ed. by E. Hatton, London, 1761), Benjamin Martin (*Decimal Arithmetick*, London, 1763; said to have been first published in 1735), Edward Hatton (*Intire System of Arithmetic*, London, 1721) and in editions of Edmund Wingate. Wingate was a London lawyer who pursued mathematics for pastime. Spending a few years in Paris, he published there in 1625[1] his *Arithmétique logarithmique*, which appeared in London in English translation in 1635.[2] Wingate was the first to carry Briggian logarithms (taken from Gunter's tables) into France. About the year 1629 he published his *Arithmetick* "in which his principal design was to obviate the difficulties which ordinarily occur in the using of logarithms: To perform this he divided his work into two books; the first he called *Natural*, and the second *Artificial Arithmetic*."[3] Subsequent editions of Wingate rested on the first of those two books, and were brought out by John Kersey, later by George Shelley, and finally by James Dodson. The work was modified so largely, that Wingate would not have known it.

[1] De Morgan, *Arith. Books;* In Maximilien Marie's, *Histoire des Sciences Mathématiques et Physiques*, Vol. III., p. 225, is given the date 1626, instead of 1625. [2] Marie, Vol. III., p. 225.

[3] James Dodson's Preface to Wingate's *Arithmetic*, London, 1760. The term "artificial numbers" for logarithms is due to Napier himself.

It is interesting to observe how writers sometimes introduced subjects of purely theoretical value into *practical* arithmetics. Perhaps authors thought that theoretical points which they themselves had mastered and found of interest, ought to excite the curiosity of their readers. Among these subjects are square-, cube-, and higher roots, continued fractions, circulating decimals, and tables of the powers of 2 up to the 144th. The last were "very useful for laying up grains of corn on the squares of a chess-board, ruining people by horseshoe bargains, and other approved problems" (De Morgan). The subject of circulating decimals was first elaborated by John Wallis (*Algebra*, Ch. 89), Leonhard Euler (*Algebra*, Book I., Ch. 12), and John Bernoulli. Circulating decimals were at one time "suffered to embarrass books on practical arithmetic, which need have no more to do with them than books on mensuration with the complete quadrature of the circle."[1] The *Decimal Arithmetic* of 1742 by John Marsh is almost entirely on this subject. As his predecessors he mentions Wallis, Jones (1706), Ward, Brown, Malcolm, Cunn, Wright.

After the great fire of London, in 1666, the business of fire insurance began to take practical shape, and in 1681 the first regular fire insurance office was opened in London. The first office in Scotland was established in 1720, the first in Germany in 1750, the first in the United States at Philadelphia in 1752, with Benjamin Franklin as one of the directors.[2] In course of time, fire insurance received some attention in English arithmetics. In 1734 the first approach to modern life insurance was made, but all members were rated alike, irrespective of age. In 1807 we have the first instance of rating "according to age and other circumstances."

It is interesting to observe that before the middle of the

[1] De Morgan, *Arith. Books*, p. 69.

[2] Article "Insurance" in the *Encyclopædia Britannica*, 9th Ed.

eighteenth century it was the custom in England to begin the legal year with the 25th of March. Not until 1752 did the counting of the new year begin with the first of January and according to the Gregorian calendar.[1] In 1752, eleven days were dropped between the 2d and 14th of September, thereby changing from "old style" to "new style."

The order in which the various subjects were treated in old arithmetics was anything but logical. The definitions are frequently given in a collection at the beginning. Dilworth develops the rules for "whole numbers," then develops the same rules for "vulgar fractions," and again for "decimal fractions." Thus he gives the "rule of three," later the "rule of three in fractions," and, again, the "rule of three direct in decimals." John Hill, in his arithmetic (edition of 1761), adds to this "the golden rule in logarithms." Fractions are taken up late. Evidently many students had no expectation of ever reaching fractions, and, for their benefit, the first part of the arithmetics embraced all the commercial rules. In the eighteenth century the practice of postponing fractions to the last became more prevalent.[2] Moreover, the treatment of this subject was usually very meagre. While the better types

[1] E. Stone in his *New Mathematical Dictionary*, London, 1743, speaks of the various early changes of the calendar and then says, "Pope Gregory XIII. pretended to reform it again, and ordered his account to be current, as it still is in all the Roman Catholick countries." Much prejudice, no doubt, lay beneath the word "pretended," and the word "still," in this connection, now causes a smile.

[2] John Kersey, in the 16th Ed. of Wingate's *Arithmetick*, London, 1735, says in his preface, "For the Ease and Benefit of those Learners that desire only so much Skill in *Arithmetick* as is useful in Accompts, Trade, and such like ordinary Employments; the Doctrine of Numbers (which, in the *First Edition*, was intermingled with Definitions and Rules concerning Broken Numbers, commonly called Fractions) is now entirely handled a-part. . . . So that now *Arithmetick* in Whole Numbers

of arithmetics, Wingate for instance, show how to find the L.C.D. in the addition of fractions, the majority of books take for the C.D. the product of the denominators. Thus, Cocker gives 8000 as the C.D. of $\frac{3}{5}$, $\frac{7}{8}$, $\frac{9}{10}$, $\frac{12}{20}$; Dilworth gives $\frac{1}{2}+\frac{7}{8}=1\frac{6}{16}$; Hatton gives $\frac{7}{12}+\frac{3}{8}=\frac{92}{96}=\frac{23}{24}$.

It was the universal custom to treat the rule of three under two distinct heads, "Rule of Three Direct" and "Rule of Three Inverse." The former embraces problems like this, "If 4 Students spend 19 Pounds, how many Pounds will 8 students spend at the same Rate of Expence?" (*Wingate.*) The inverse rule treats questions of this sort, "If 8 horses will be maintained 12 Days with a certain Quantity of Provender, How many Days will the same Quantity maintain 16 Horses?" (*Wingate.*) In problems like the former the correct answer could be gotten by taking the three numbers in order as the first three terms of a proportion. In Wingate's notation, we would have, If

Students	Pounds	Students
4	19	8.

But in the second example "you cannot say here in a direct proportion (as before in the Rule of Three Direct) as 8 to 16, so is 12 to another Number which ought to be in that Case as great again as 12; but contrariwise by an *inverted Proportion*, beginning with the last Term first; as 16 is to 8, so is 12 to another Number." (*Wingate*, 1735, p. 57.) This brings out very clearly the appropriateness of the term "Rule of Three *inverse*." As all problems were classified by authors under two distinct heads, the direct rule and the inverse rule, the pupil could get correct answers by a purely mechanical process, without being worried by the question, under which rule

is plainly and fully handled before any Ent'rance be made into the craggy Paths of Fractions, at the Sight of which some Learners are so discouraged, that they make a stand, and cry out, *non plus ultra*, There's no Progress farther."

does the problem come? Thus that part of the subject which taxes to the utmost the skill of the modern teacher of proportion, was formerly disposed of in an easy manner. No heed was paid to mental discipline. Nor did authors care what the pupil would do with a problem, when he was not told beforehand to what rule it belonged.

Beginning with the time of Cocker, all demonstrations are carefully omitted. The only proofs known to Dilworth are of this kind, "Multiplication and division prove each other." The only evidence we could find that John Hill was aware of the existence of such things as mathematical demonstrations lies in the following passage, "So $\frac{1}{2}$ multiplied by $\frac{1}{2}$ becomes $\frac{1}{4}$. See this demonstrated in Mr. Leyburn's Cursus Mathematicus, page 38." What a contrast between this and our quotation on fractions from Tonstall! The practice of referring to other works was more common in arithmetics then than now. Cocker refers to Kersey's *Appendix*, to Wingate's *Arithmetic*, to Pitiscus's *Trigonometria*, and to Oughtred's *Clavis* for the proof of the rule of Double Position. Cocker repeatedly gives on the margin Latin quotations from the *Clavis*, from Alsted's *Mathematics*, or Gemma-Frisius's *Methodus Facilis* (Wittenberg, 1548).

Toward the end of the eighteenth century demonstrations begin to appear, in the better books, but they are often placed at the foot of the page, beneath a horizontal line, the rules and examples being above the line. By this arrangement the author's conscience was appeased, while the teacher and pupil who did not care for proofs were least annoyed by their presence and could easily skip them. Moreover, the proofs and explanations were not adapted to the young mind; there was no trace of object-teaching; the presentation was too abstract. True reform, both in England and America, began only with the introduction of Pestalozzian ideas.

Mental arithmetic received no attention in England before the present century, but in Germany it was introduced during the second half of the eighteenth century.[1]

Causes which Checked the Growth of Demonstrative Arithmetic in England

Before the Reformation there was little or nothing accomplished in the way of public education in England. In the monasteries some instruction was given by monks, but we have no evidence that any branch of mathematics was taught to the youth.[2] In 1393 was established the celebrated "public school," named *Winchester*, and in 1440, *Eton*. In the sixteenth century, on the suppression of the monasteries, schools were founded in considerable numbers — *The Merchant Taylors' School*, *Christ Hospital*, *Rugby*, *Harrow*, and others — which in England monopolize the title "public schools" and for centuries have served for the education of the sons mainly of the nobility and gentry.[3] In these schools the ancient classics were the almost exclusive subjects of study; mathematical teaching was unknown there. Perhaps the demands of every-day life forced upon the boys a knowledge of counting and of the very simplest computations, but we are safe in saying that, before the close of the last century, the ordinary boy of England's famous public schools

[1] Unger, p. 168.

[2] Some idea of the state of arithmetical knowledge may be gathered from an ancient custom at Shrewsbury, where a person was deemed of age when he knew how to count up to twelve pence. (Year-Books Edw. I., XX.–I. Ed. Horwood, p. 220). See Tylor's *Primitive Culture*, New York, 1889, Vol. I., p. 242.

[3] Consult Isaac Sharpless, *English Education*, New York, 1892; H. F. Reddall, *School-boy Life in Merrie England*, New York, 1891; John Timbs, *School-Days of Eminent Men*, London.

could not divide 2021 by 43, though such problems had been performed centuries before according to the teaching of Brahmagupta and Bhaskara by boys brought up on the far-off banks of the Ganges. It has been reported that Charles XII. of Sweden considered a man ignorant of arithmetic but half a man. Such was not the sentiment among English gentlemen. Not only was arithmetic unstudied by them, but considered beneath their notice. If we are safe in following Timbs's account of a book of 1622, entitled *Peacham's Compleat Gentleman*, which enumerates subjects at that time among the *becoming* accomplishments of an English man of rank, then it appears that the elements of astronomy, geometry, and mechanics were studies beginning to demand a gentleman's attention, while arithmetic still remained untouched.[1] Listen to another writer, Edmund Wells. In his *Young Gentleman's Course in Mathematicks*, London, 1714, this able author aims to provide for gentlemanly education as opposed to that of "the meaner part of mankind."[2] He expects those whom God has relieved from the necessity of working, to exercise their faculties to his greater glory. But they must not "be so Brisk and Airy, as to think, that the knowing how to cast Accompt is requisite only for such Underlings as Shop-keepers or Tradesmen," and if only for the sake of taking care of themselves, "no gentleman ought to think Arithmetick below Him that do's not think an Estate below Him." All the information we could find respecting the education of the upper classes points to the conclusion that arithmetic was neglected, and that De Morgan[3] was right in his statement that as late as the eighteenth century there could have been no such thing as a teacher of arithmetic in schools like Eton. In 1750, Warren Hastings,

[1] Timbs, *School-Days*, p. 101.
[2] De Morgan, *Arith. Books*, p. 64.
[3] *Arith. Books*, p. 76.

who had been attending Westminster, was put into a commercial school, that he might study arithmetic and book-keeping before sailing for Bengal.

At the universities little was done in mathematics before the middle of the seventeenth century. It would seem that Tonstall's work was used at Cambridge about 1550, but in 1570, during the reign of Queen Elizabeth, fresh statutes were given, excluding all mathematics from the course of undergraduates, presumably because this study pertained to practical life, and could, therefore, have no claim to attention in a university.[1]

The commercial element in the arithmetics and algebras of early times was certainly very strong. Observe how the Arab Muḥammed ibn Mûsâ and the Italian writers discourse on questions of money, partnership, and legacies. Significant is the fact that the earliest English algebra is dedicated by Recorde to the company of merchant adventurers trading to Moscow.[2] Wright, in his English edition of Napier's logarithms, likewise appeals to the commercial classes: "To the Right Honourable And Right Worshipfvll Company Of Merchants of London trading to the East-Indies, Samvel Wright wisheth all prosperitie in this life and happinesse in the life to come."[3] It is also significant that the first headmaster of the Merchant Taylors' School, a reformer with views in advance of his age, in a book of 1581 speaks of mathematical instruction, thinks that a few of the most earnest and gifted students might hope to attain a knowledge of geometry and arithmetic from Euclid's *Elements*, but fails to notice Recorde's arithmetic, which since 1540 had appeared in edition after edition.[4] Clas-

[1] Ball, *Mathematics at Cambridge*, p. 13.

[2] G. Heppel in 19th *General Report* (1893) of the A. I. G. T., pp. 26 and 27.

[3] Napier's *Construction* (Macdonald's Ed.), p. 145.

[4] G. Heppel, *op. cit.*, p. 28.

sical men were evidently not in touch with the new in the mathematics of that time.

This scorn and ignorance of the art of computation by all but commercial classes is seen in Germany as well as England. Kästner[1] speaks of it in connection with German Latin schools; Unger refers to it repeatedly.[2]

It was not before the present century that arithmetic and other branches of mathematics found admission into England's public schools. At Harrow "vulgar fractions, Euclid, geography, and modern history were first studied" in 1829.[3] At the Merchant Taylors' School "mathematics, writing, and arithmetic were added in 1829."[4] At Eton "mathematics was not compulsory till 1851."[5] The movement against the exclusive classicism of the schools was led by Dr. Thomas Arnold of Rugby, the father of Matthew Arnold. Dr. Arnold favoured the introduction of mathematics, science, history, and politics.

Since the art of calculation was no more considered a part of a liberal education than was the art of shoe-making, it is natural to find the study of arithmetic relegated to the commercial schools. The poor boy sometimes studied it; the rich boy did not need it. In Latin schools it was unknown, but in schools for the poor it was sometimes taught; for example, in a "free grammar school founded by a grocer of London in 1553 for thirty 'of the poorest men's sons' of Guilford, to be taught to read and write English and cast accounts perfectly, so that they should be fitted for apprentices."[6] That a science, ignored as a mental discipline, and studied merely as an aid to material gain, should fail to receive fuller development, is not strange. It was, in fact, very natural that it

[1] *Geschichte*, III., p. 429.
[2] See pp. 5, 24, 112, 140, 144.
[3] Reddall, p. 228.
[4] Timbs, p. 84.
[5] Sharpless, p. 144.
[6] Timbs, p. 83.

should entirely discard those features belonging more properly to a science and assume the form of an art. So arithmetic reduced itself to a mere collection of rules. The ancient Carthaginians, like the English, studied arithmetic, but did not develop it as a science. "It is a beautiful testimony to the quality of the Greek mind that Plato and others assign as a cause of the low state of arithmetic and mathematics among the Phœnicians . . . the want of free and disinterested investigation."[1]

It will be observed that during the period under consideration, the best English mathematical minds did not make their influence felt in arithmetic. The best intellects held aloof from the elementary teacher; arithmetical texts were written by men of limited education. During the seventeenth and eighteenth centuries England had no Tonstall and Recorde, no Stifel and Regiomontanus, no Pacioli and Tartaglia, to compose her arithmetical books. To be sure, she had her Wallis, her Newton, Cotes, Hook, Taylor, Maclaurin, De Moivre, but they wrote no books for elementary schools; their influence on arithmetical teaching was naught. Contrast this period with the present century. Think of the pains taken by Augustus De Morgan to reform elementary mathematical instruction. The man who could write a brilliant work on the Calculus, who could make new discoveries in advanced algebra, series, and in logic, was the man who translated Bourdon's arithmetic from the French, composed an arithmetic and elementary algebra for younger students, and endeavoured to simplify, without loss of rigour, the Euclidean geometry. Again, run over the list of members of the "Association for the Improvement of Geometrical Teaching," and of the Committee of the British Association on geometrical

[1] J. K. F. Rosenkranz, *Philosophy of Education*, translated by A. C. Brackett, New York, 1888, p. 215.

teaching, and you will find in it England's most brilliant mathematicians of our time.

The imperfect interchange of ideas between writers on advanced and those on elementary subjects is exhibited in the mathematical works of John Ward of Chester. He published in 1695 a *Compendium of Algebra*, and in 1706 his *Young Mathematician's Guide*. The latter work appeared in the 12th edition in 1771, was widely read in Great Britain, and well approved in the universities of England, Scotland, and Ireland. It was once a favourite text-book in American colleges, being used as early as 1737 at Harvard College, and as late as 1787 at Yale and Dartmouth. In 475 pages, the book covered the subjects of Arithmetic, Algebra, Geometry, Conic Sections, and Arithmetic of Infinites, giving, of course, the mere rudiments of each.

Ward shows how to raise a binomial to positive integral powers "without the trouble of continued involution" and remarks that when he published this method in his *Compendium of Algebra* he thought that he was the first inventor of it, but that since he has found in Wallis's *History of Algebra* "that the learned Sir Isaac Newton had discovered it long before." It took a quarter of a century for the news of Newton's binomial discovery to reach John Ward. Moreover, it looks very odd to see in Ward's *Guide* of 1771 — over a century *after* Newton's discovery of fluxions — a sort of integral calculus, such as was employed by Wallis, Cavalieri, Fermat, and Roberval *before* the invention of fluxions.

It is difficult to discover a time when in any civilized country advanced and elementary writers on mathematics were *more* thoroughly out of touch with each other than in England during the two centuries and a half preceding 1800. We can think of no other instance in science in which the failure of the best minds to influence the average has led to such

lamentable results. In recent times we have heard it deplored that in some countries practical chemistry does not avail itself of the results of theoretical chemistry. Fears have been expressed of a coming schism between applied higher mathematics and theoretical higher mathematics. But thus far these evils appear insignificant, as compared with the widespread repression and destruction of demonstrative arithmetic, arising from the failure of higher minds to guide the rank and file. Neglected by the great thinkers of the day, scorned by the people of rank, urged onward by considerations of purely material gain, arithmetical writers in England (as also in Germany and France) were led into a course which for centuries blotted the pages of educational history.

In those days English and German boys often prepared for a business career by attending schools for writing and arithmetic. During the Middle Ages and also long after the invention of printing, the art of writing was held in high esteem. In writing schools much attention was given to fancy writing. The schools embracing both subjects are always named schools for "writing and arithmetic," never "arithmetic and writing." The teacher was called "writing-master and arithmetician." Cocker was skilful with the pen, and wrote many more books on calligraphy than on arithmetic. As to the quality of the teachers, Peacham in his *Compleat Gentleman* (1622) stigmatizes the schoolmasters in his own day as rough and even barbarous to their pupils. Domestic tutors he represents as still worse.[1] The following from an arithmetic of William Webster, London, 1740, gives an idea of the origin of many commercial schools.[2] "When a Man has tried all Shifts, and still failed, if he can but scratch out anything like a fair *Character*, tho never so stiff and

[1] Timbs, p. 101 [2] De Morgan, *Arith. Books*, p. 69.

unnatural and has got but *Arithmetick* enough in his Head to compute the Minutes in a Year, or the Inches in a Mile, he makes his last Recourse to a Garret, and with the Painter's Help, sets up for a Teacher of *Writing* and *Arithmetick;* where, by the Bait of low Prices, he perhaps gathers a Number of Scholars." No doubt in previous centuries, as in all times, there were some good teachers, but the large mass of schoolmasters in England, Germany, and the American colonies were of the type described in the above extract. Some of the more ambitious and successful of these teachers wrote the arithmetics for the schools. No wonder that arithmetical authorship and teaching were at a low ebb.

To summarize, the causes which checked the growth of demonstrative arithmetic are as follows:

(1) Arithmetic was not studied for its own sake, nor valued for the mental discipline which it affords, and was, consequently, learned only by the commercial classes, because of the material gain derived from a knowledge of arithmetical rules.

(2) The best minds failed to influence and guide the average minds in arithmetical authorship.

Reforms in Arithmetical Teaching

A great reform in elementary education was initiated in Switzerland and Germany at the beginnning of the nineteenth century. Pestalozzi emphasized in all instruction the necessity of object-teaching. "Of the instruction at Yverdun, the most successful in the opinion of those who visited the school, was the instruction in arithmetic. The children are described as performing with great rapidity very difficult tasks in head-calculation. Pestalozzi based his method here, as in other subjects, on the principle that the individual should be brought to the knowledge by the road similar to that which

the whole race had used in founding the science. Actual counting of things preceded the first Cocker, as actual measurement of land preceded the original Euclid. The child must be taught to count things and to find out the various processes experimentally in the concrete, before he is given any abstract rule, or is put to abstract exercises. This plan is now adopted in German schools, and many ingenious contrivances have been introduced by which the combinations of things can be presented to the children's sight."[1] Much remained to be done by the followers of Pestalozzi in the way of practical realization of his ideas. Moreover, a readjustment of the course of instruction was needed, for Pestalozzi quite ignored those parts of arithmetic which are applied in every-day life. Until about 1840 Pestalozzi's notions were followed more or less closely. In 1842 appeared A. W. Grube's *Leitfaden*, which was based on Pestalozzi's idea of object-teaching, but, instead of taking up addition, subtraction, multiplication, and division in the order as here given, advocated the exclusion of the larger numbers at the start, and the teaching of all four processes in connection with the first circle of numbers (say, the numbers 1 to 10) before proceeding to a larger circle. For a time Grube's method was tried in Germany, but it soon met with determined opposition. The experience of both teachers and students seems to be that, to reach satisfactory results, an extraordinary expenditure of energy is demanded. In the language of physics, Grube's method seems to be an engine having a low efficiency.[2]

Into conservative England Pestalozzian ideas found tardy ad-

[1] R. H. Quick, *Educational Reformers*, 1879, p. 191.

[2] For the history of arithmetical teaching in Germany during the nineteenth century, see Unger, pp. 175–233. For a discussion of the psychological bearing of Grube's method, see McLellan and Dewey, *Psychology of Number*, 1895, p. 80 *et seq.*

mittance. At the time when De Morgan began to write (about 1830) arithmetical teaching had not risen far above the level of the eighteenth century. In more recent time the teaching of arithmetic has been a subject of discussion by the "Association for the Improvement of Geometrical Teaching."[1] Among the subjects under consideration have been the multiplication and division of concrete quantities, the approximate multiplication of decimals (by leaving off the tail of the product), and modification in the processes of subtraction, multiplication, and division. The new modes of subtraction and division are called in Germany the "Austrian methods," because the Austrians were the first to adopt them. In England this mode of division goes by the name of "Italian method."[2]

The "Austrian" method of subtraction is simply this: It shall be performed in the same way as "change" is given in a store by adding from the lower to the higher instead of passing from the higher to the lower by mental subtraction. Thus, in $76 - 49$, say "nine and *seven* are sixteen, five and *two* are seven." The practice of adding 1 to the 4, instead of subtracting 1 from 7, was quite common during the Renaissance. It was used, for instance, by Maximus Planudes Georg Peuerbach, and Adam Riese. In Adam Riese's works there is also an approach to the determination of the difference by figuring from the subtrahend upwards. In the above example, he would subtract 9 from 10 and add the remainder 1 to the minuend 6, thus obtaining the answer 7.[3] A recent American text-book on algebra explains subtraction on the principle of the "Austrian method."

[1] See the *General Reports* since 1888, especially those for 1892 and 1893.

[2] The name "Italian method" was traced by Mr. Langley back to an English arithmetic of 1730. See *General Report* of A. I. G. T., 1892, p. 34.

[3] Arno Sadowski, *Die österreichische Rechenmethode in pädagogischer und historischer Beleuchtung*, Königsberg, 1892, p. 13.

In multiplication, the recommendation is to begin, as in algebra, with the figure of highest denomination in the multiplier. The great advantage of this shows itself in decimal multiplication in case we desire only an approximate answer correct to, say, three or five decimal places. We have seen that this method was much used by the Florentines and was called by Pacioli "by the little castle." Of the various multiplication processes of the Renaissance, the fittest failed to survive.[1] The process now advocated was taught by Nicholas Pike[2] in the following example: "It is required to multiply 56.7534916 by 5.376928, and to retain only five places of decimals in the product."

56.7534 916
82 9673.5

28376746 . .
1702605 . . .
397274 . . . .
34052 . . . . .
5108 . . . . . .
113 . . . . . .
45 . . . . . .

305.15943

The "Austrian" or "Italian" method of division simply calls for the multiplication of the divisor and the subtraction from the dividend simultaneously, so that only the remainder is written down on paper. See our illustration. It is doubtful whether this method is preferable to our old method, except for naturally rapid computors. We fear that the slow computor saves paper by it, at the expense of mental energy. The "Austrian method of division" is

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[1] This process was favoured by Lagrange. He says: "But nothing compels us to begin with the right side of the multiplier; we may as well begin with the left side, and I truly fail to see why this method is not preferred, for it has the great advantage of giving the places of highest value first; in the multiplication of large numbers we are often interested most in the highest places." See H. Niedermüller, *Lagrange's Mathematische Elementarvorlesungen.* Deutsche Separatausgabe, Leipzig, 1880, p. 23. This publication, giving five lectures on arithmetic and algebra, delivered by the great Lagrange at the Normal School in Paris, in 1795, is very interesting for several reasons.

[2] N. Pike, *Arithmetic,* abridged for the use of schools, 3d Ed., Worcester, 1798, p. 92.

not entirely new; in the "galley" or "scratch" method, the partial products were not written down, but at once subtracted, and only the remainder noted. In principle, the "Austrian method" was practised by the Hindus, of whose process the "scratch" method is a graphical representation.[1]

Arithmetic in the United States

Weights and Measures. — The weights and measures introduced into the United States were derived from the English. "It was from the standards of the exchequer that all the weights and measures of the United States were derived, until Congress fixed the standard. Louisiana at first recognized standards derived from the French, but in 1814 the United States revenue standards were established by law."[2] The actual standards used in the several states and in the custom-houses were, however, found to be very inaccurate. In the construction of accurate standards for American use, our Government engaged the services of *Ferdinand Rudolph Hassler*, a Swiss by birth and training, and a skilful experimentalist.[3] The work of actual construction was begun in 1835.[4] In 1836, carefully constructed standards were distributed to the custom-houses and furnished the means of uniformity in the collection of the customs. Moreover, accurate standards were distributed by the general Government to the

[1] For further information regarding the "Austrian" method of subtraction and division, see SADOWSKI, *op. cit.*; UNGER, pp. 213–218.

[2] *Report of the Secretary of the Treasury on the Construction and Distribution of Weights and Measures. Ex. Doc. No. 27*, 1857, p. 36.

[3] Consult the translation from the German of Memoirs of Ferdinand Rudolph Hassler, by EMIL ZSCHOKKE, published in Aarau, Switzerland, 1877; with supplementary documents, published at Nice, 1882. See also *Teach. and Hist. of Math. in the U. S.*, pp. 286–289.

[4] *Ex. Doc. No. 84* (Report by A. D. Bache for 1846–47), p. 2.

various States, with the view of securing greater uniformity. It was recommended that each State prepare for them a fire-proof room and place them "under the charge of some scientific person who would attend to their use and safe-keeping."

In 1866 Congress authorized the use of the Metric System in the United States, but unfortunately stopped here and allowed the nations of Continental Europe to advance far beyond us by their adoption of the Metric System to the exclusion of all older systems.

In the currency of the American colonies there existed great diversity and confusion. "At the time of the adoption of our decimal currency by Congress, in 1786, the *colonial currency* or *bills of credit*, issued by the colonies, had depreciated in value, and this depreciation, being unequal in the different colonies, gave rise to the different values of the State currencies."[1] Inasmuch as all our early arithmetics were "practical" arithmetics, they, of course, gave rules for the "reduction of coin." Thus Pike's *Arithmetic*[2] devotes twenty-two pages to the statement and illustration of rules for reducing "Newhampshire, Massachusetts, Rhodeisland, Connecticut, and Virginia currency" (1) to "Federal Money," (2) to "Newyork and Northcarolina currency," (3) to "Pennsylvania, Newjersey, Delaware, and Maryland currency, . . ." (6) to "Irish money," (7) to "Canada and Novascotia currency," (8) to "Livres Tournois," (9) to "Spanish milled dollars." Then follow rules for reducing Federal Money to "Newengland and Virginia currency," etc. It is easy to see how a large share of the pupil's time was absorbed in the mastery of these rules. The chapters on reduction of coins, on duodecimals, alligation, etc.,

[1] Robinson, *Progressive Higher Arithmetic*, 1874, p. 190.

[2] *The New and Complete System of Arithmetic*, abridged for the use of schools, 3d Ed. 1798, Worcester, pp. 117–139.

give evidence of the homage that Education was forced to pay to Practical Life, at the sacrifice of matter better fitted to develop the mind of youth. With a view of supplying the information needed by merchants in business, arithmetics discussed such subjects as the United States Securities, the various rules adopted by the United States, and by the State governments on partial payments.

Authors and Books. — The first arithmetics used in the American colonies were English works: Cocker, Hodder, Dilworth, "George Fisher" (Mrs. Slack), Daniel Fenning.[1] The earliest arithmetic written and printed in America appeared anonymously in Boston in 1729. Though a work of considerable merit, it seems to have been used very little; in early records we have found no reference to it; fifty years later, at the publication of Pike's *Arithmetic*, the former work was completely forgotten, and Pike's was declared to be the earliest American arithmetic. Of the 1729 publication there are two copies in the Harvard Library and one in the Congressional Library.[2] In Appleton's *Cyclopædia of American Biography* its authorship is ascribed without reserve to Isaac Greenwood, then professor of mathematics at Harvard College, but on the title-page of one of the copies in the Harvard Library, is written the following: "Supposed that Sam! Greenwood was the author thereof, by others said to be by Isaac Greenwood." In 1788 appeared at Newburyport the *New and Complete System of Arithmetic* by Nicholas Pike (1743–1819), a graduate of Harvard College.[3] It was intended for advanced schools, and contained, besides the ordinary subjects of that time, logarithms, trigonometry, algebra, and conic sections; but these latter subjects were so briefly treated as to possess little value. In the "Abridgment for the Use of Schools,"

[1] See *Teach. and Hist. of Math. in the U. S.*, pp. 12–16.
[2] Ibidem, p. 14. [3] Ibidem, pp. 45, 46.

which was brought out at Worcester in 1793, the larger work is spoken of in the preface as "now used as a classical book in all the Newengland Universities." A recent writer[1] makes Pike responsible for all the abuses in arithmetical teaching that prevailed in early American schools. To us this condemnation of Pike seems wholly unjust. It is unmerited, even if we admit that Pike was in no sense a reformer among arithmetical authors. Most of the evils in question have a far remoter origin than the time of Pike. Our author is fully up to the standard of English works of that date. He can no more be blamed by us for giving the aliquot parts of pounds and shillings, for stating rules for "tare and trett," for discussing the "reduction of coins," than the future historian can blame works of the present time for treating of such atrocious relations as that 3 ft. = 1 yd., $5\frac{1}{2}$ yds. = 1 rd., $30\frac{1}{4}$ sq. yds. = 1 sq. rd., etc. So long as this free and independent people chooses to be tied down to such relics of barbarism, the arithmatician cannot do otherwise than supply the means of acquiring the precious knowledge.

At the beginning of the nineteenth century there were three "great arithmeticians" in the United States: Nicholas Pike, Daniel Adams, and Nathan Daboll. The arithmetics of Adams (1801) and of Daboll (1800) paid more attention than that of Pike to Federal Money. Peter Parley tells us that in consequence of the general use, for over a century, of Dilworth in American schools, pounds, shillings, and pence were classical, and dollars and cents vulgar for several succeeding generations. "I would not give a penny for it" was genteel; "I would not give a cent for it" was plebeian.

Reform in arithmetical teaching in the United States did not begin until the publication by Warren Colburn, in 1821,

[1] GEORGE H. MARTIN, *The Evolution of the Massachusetts Public School System*, p. 102.

of the *Intellectual Arithmetic*.[1] This was the first fruit of Pestalozzian ideas on American soil. Like Pestalozzi, Colburn's great success lay in the treatment of mental arithmetic. The success of this little book was extraordinary. But American teachers in Colburn's time, and long after, never quite succeeded in successfully engrafting Pestalozzian principles on written arithmetic. Too much time was assigned to arithmetic in schools. There was too little object-teaching; either too much abstruse reasoning,[2] or no reasoning at all; too little attention to the art of rapid and accurate computation; too much attention to the technicalities of commercial arithmetic. During the last ten years, however, desirable reforms have been introduced.[3]

"Pleasant and Diverting Questions"

In English and American editions of Dilworth, as also in Daniel Adams's *Scholar's Arithmetic*[4] we find a curious collection of "Pleasant and diverting questions." We have all heard of the farmer, who, having a fox, a goose, and a peck of corn, wished to cross a river; but, being able to carry only one at a time, was confounded as to how he should take

[1] "Warren Colburn's *First Lessons* have been abused by being put in the hands of children too early, and has been productive of almost as much harm as good." — Rev. Thomas Hill, *The True Order of Studies*, 1876, p. 42.

[2] "The teacher who has been accustomed to the modern erroneous method of teaching a child to reason out his processes from the beginning may be assured this method of gaining facility in the operations, before attempting to explain them, is the method of Nature; and that it is not only much pleasanter to the child, but that it will make a better mathematician of him." — T. Hill, *op. cit.*, p. 45.

[3] For a more detailed history of arithmetical teaching, see *Teach. and Hist. of Math. in the U. S.*

[4] Seventh Ed., Montpelier, Vt., 1812, p. 210.

them across so that the fox should not devour the goose, nor the goose the corn. Who has not been entertained by the problem, how three jealous husbands with their wives may cross a river in a boat holding only two, so that none of the three wives shall be found in company of one or two men unless her husband be present? Who has not attempted to place nine digits in a square so that any three figures in a line may make just 15? None of us, perhaps, at first suspected the great antiquity of these apparently new-born creatures of fancy. Some of these puzzles are taken by Dilworth from Kersey's edition of Wingate. Kersey refers the reader to "the most ingenious" *Gaspar Bachet de Méziriac* in his little book, *Problèmes plaisants et délectables qui se font par les nombres* (Lyons, 1624), which is still largely read. The first of the above puzzles was probably known to Charlemagne, for it appears in Alcuin's (?) *Propositiones ad acuendos juvenes*, in the modified version of the wolf, goat, and cabbage puzzle. The three jealous husbands and their wives were known to Tartaglia, who also proposes the same question with four husbands and four wives.[1] We take these to be modified and improved versions of the first problem. The three jealous husbands have been traced back to a MS. of the thirteenth century, which represents two German youths, Firri and Tyrri, proposing problems to each other.[2] The MS. contains also the following: Firri says: "There were three brothers in Cologne, having nine vessels of wine. The first vessel contained one quart (amam), the second 2, the third 3, the fourth 4, the fifth 5, the sixth 6, the seventh 7, the eighth 8, the ninth 9. Divide the wine equally among the three brothers, without mixing the contents of the vessels." This

[1] Peacock, p. 473.

[2] Dr. S. Günther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter*, Berlin, 1887, p. 35.

question is closely related to the third problem given above, since it gives rise to the following magic square demanded by that problem.

Magic squares were known to the Arabs and, perhaps, to the Hindus. To the Byzantine writer, Moschopulus, who lived in Constantinople in the early part of the fifteenth century, appears to be due the introduction into Europe of these curious and ingenious products of mathematical thought. Mediæval astrologers believed them to possess mystical properties and when engraved on silver plate to be a charm against plague.[1] The first complete magic square which has been discovered in the Occident is that of the German painter, Albrecht Dürer, found on his celebrated wood-engraving, "Melancholia."

2	7	6
9	5	1
4	3	8

Of interest is the following problem, given in Kersey's Wingate: "15 Christians and 15 Turks, being at sea in one and the same ship in a terrible storm, and the pilot declaring a necessity of casting the one half of those persons into the sea, that the rest might be saved; they all agreed, that the persons to be cast away should be set out by lot after this manner, viz. the 30 persons should be placed in a round form like a ring, and then beginning to count at one of the passengers, and proceeding circularly, every ninth person should be cast into the sea, until of the 30 persons there remained only 15. The question is, how those 30 persons ought to be placed, that the lot might infallibly fall upon the 15 Turks and not upon any of the 15 Christians?" Kersey lets the letters *a*, *e*, *i*, *o*, *u* stand, respectively, for 1, 2, 3, 4, 5, and gives the verse

> From numbers' aid and art,
> Never will fame depart.

[1] For the history of Magic Squares, see GÜNTHER, *Vermischte Untersuchungen*, Ch. IV. Their theory is developed in the article "Magic Squares" in JOHNSON'S *Universal Cyclopædia*.

The vowels in these lines, taken in order, indicate alternately the number of Christians and Turks to be placed together; *i.e.*, take $o = 4$ Christians, then $u = 5$ Turks, then $e = 2$ Christians, etc. Bachet de Méziriac, Tartaglia, and Cardan give each different verses to represent the rule. According to a story related by Hegesippus,[1] the famous historian Josephus, the Jew, while in a cave with 40 of his countrymen, who had fled from the conquering Romans at the siege of Jotapata, preserved his life by an artifice like the above. Rather than be taken prisoners, his countrymen resolved to kill one another. Josephus prevailed upon them to proceed by lot and managed it so that he and one companion remained. Both agreed to live.

The problem of the 15 Christians and 15 Turks has been called by Cardan *Ludus Joseph*, or Joseph's Play. It has been found in a French work of 1484 written by Nicolas Chuquet[2] and in MSS. of the twelfth, eleventh, and tenth centuries.[3] Daniel Adams gives in his arithmetic the following stanza:

"As I was going to St. Ives,
I met seven wives.
Every wife had seven sacks;
Every sack had seven cats;
Every cat had seven kits:
Kits, cats, sacks, and wives,
How many were going to St. Ives?"

Compare this with Fibonaci's "Seven old women go to Rome," etc., and with the problem in the Ahmes papyrus, and we perceive that of all problems in "mathematical recreations" this is the oldest.

Pleasant and diverting questions were introduced into some English arithmetics of the latter part of the seventeenth and

[1] *De Bello Judaico*, etc., III., Ch. 15. [2] CANTOR, Vol. II., p. 362.
[3] M. CURTZE, in *Biblio. Mathem.*, 1894, p. 116 and 1895, pp. 34–36.

of the eighteenth centuries. In Germany this subject found entrance into arithmetic during the sixteenth century. Its aim was to make arithmetic more attractive. In the seventeenth century a considerable number of German books were wholly devoted to this subject.[1]

Certain writers have found amusement in speculations over the origin of the $. Over a dozen different theories have been advanced, but no serious effort was made to verify any of them. A careful study of manuscripts, made recently,[2] has shown that the $ is the lineal descendant of the Spanish abbreviation p^s for "pesos," that the change from the florescent p^s to $ was made about 1775 by English-Americans who came in business relations with Spanish-Americans. The earliest known printed $ occurs in an arithmetic, Chauncey Lee's *American Accomptant*, published in 1797 at Lansingburgh.

[1] WILDERMUTH.

[2] See F. CAJORI in the *Popular Science Monthly*, December, 1912, pp. 521–530; *Science*, N. S., Vol. 38, 1913, pp. 848–850.

ALGEBRA

The Renaissance

One of the great steps in the development of algebra during the sixteenth century was the algebraic solution of cubic equations. The honour of this remarkable feat belongs to the Italians.[1] The first successful attack upon cubic equations was made by *Scipio Ferro* (died in 1526), professor of mathematics at Bologna. He solved cubics of the form $x^3 + mx = n$, but nothing more is known of his solution than that he taught it to his pupil *Floridus* in 1505. It was the practice in those days and during centuries afterwards for teachers to keep secret their discoveries or their new methods of treatment, in order that pupils might not acquire this knowledge, except at their own schools, or in order to secure an advantage over rival mathematicians by proposing problems beyond their reach. This practice gave rise to many disputes on the priority of inventions. One of the most famous of these quarrels arose in connection with the discovery of cubics, between Tartaglia and Cardan. In 1530 one Colla proposed to Tartaglia several problems, one leading to the equation $x^3 + px^2 = q$. The latter found an imperfect method of resolving this, made known his success, but kept

[1] The geometric solution had been given previously by the Arabs.

his solution secret. This led Ferro's pupil Floridus to proclaim his knowledge of how to solve $x^3 + mx = n$. Tartaglia challenged him to a public contest to take place Feb. 22, 1535. Meanwhile he worked hard, attempting to solve other cases of cubic equations, and finally succeeded, ten days before the appointed date, in mastering the case $x^3 = mx + n$. At the contest each man proposed 30 problems. The one who should be able to solve the greater number within fifty days was to be the victor. Tartaglia solved his rival's problems in two hours; Floridus could not solve any of Tartaglia's. Thenceforth Tartaglia studied cubic equations with a will, and in 1541 he was in possession of a general solution. His fame began to spread throughout Italy. It is curious to see what interest the enlightened public took in contests of this sort. A mathematician was honoured and admired for his ability. Tartaglia declined to make known his method, for it was his aim to write a large work on algebra, of which the solution of cubics should be the crowning feature. But a scholar of Milan, named *Hieronimo Cardano* (1501–1576), after many solicitations and the most solemn promises of secrecy, succeeded in obtaining from Tartaglia the method. Cardan thereupon inserted it in a mathematical work, the *Ars Magna*, then in preparation, which he published in 1545. This breach of promise almost drove Tartaglia mad. His first step was to write a history of his invention, but to completely annihilate Cardan, he challenged him and his pupil Ferrari to a contest. Tartaglia excelled in his power of solving problems, but was treated unfairly. The final outcome of all this was that the man to whom we owe the chief contribution to algebra made in the sixteenth century was forgotten, and the discovery in question went by the name of Cardan's solution. Cardan was a good mathematician, but the association of his name with the discovery of the solution of cubics is a

gross historical error and a great injustice to the genius of Tartaglia.

The success in resolving cubics incited mathematicians to extraordinary efforts toward the solution of equations of higher degrees. The solution of equations of the fourth degree was effected by Cardan's pupil, *Lodovico Ferrari*. Cardan had the pleasure of publishing this brilliant discovery in the *Ars Magna* of 1545. Ferrari's solution is sometimes ascribed to Bombelli, who is no more the discoverer of it than Cardan is of the solution called by his name. For the next three centuries algebraists made innumerable attempts to discover algebraic solutions of equations of higher degree than the fourth. It is probably no great exaggeration to say that every ambitious young mathematician sooner or later tried his skill in this direction. At last the suspicion arose that this problem, like the ancient ones of the quadrature of the circle, duplication of the cube, and trisection of an angle, did not admit of the kind of solution sought. To be sure, particular forms of equations of higher degrees could be solved satisfactorily. For instance, if the coefficients are all numbers, some method like that of Vieta, Newton, or Horner, always enables the computor to approximate to the numerical values of the roots. But suppose the coefficients are letters which may stand for any number whatever, and that no relation is assumed to exist between these coefficients, then the problem assumes more formidable aspects. Finally it occurred to a few mathematicians that it might be worth while to try to prove the impossibility of solving the quintic algebraically; that is, by radicals. Thus, an Italian physician, *Paolo Ruffini* (1765–1822), printed proofs of their insolvability,[1] but these proofs were declared inconclusive by his countryman Malfatti.

[1] See H. Burkhardt, "Die Anfänge der Gruppentheorie und Paolo Ruffini" in *Zeitschr. f. Math. u. Physik*, Suppl., 1892.

Later a brilliant young Norwegian, *Niels Henrik Abel* (1802–1829), succeeded in establishing by rigorous proof that the general algebraic equation of the fifth or of higher degrees cannot be solved by radicals.[1] A modification of Abel's proof was given by Wantzel.[2]

Returning to the Renaissance it is interesting to observe, that Cardan in his works takes notice of negative roots of an equation (calling them *fictitious*, while the positive roots are called *real*), and discovers all three roots of certain numerical cubics (no more than two roots having ever before been found in any equation). While in his earlier writings he rejects imaginary roots as impossible, in the *Ars Magna* he exhibits great boldness of thought in solving the problem, to divide 10 into two parts whose product is 40, by finding the answers $5+\sqrt{-15}$ and $5-\sqrt{-15}$, and then multiplying them together, obtaining $25+15=40$.[3] Here for the first time we see a decided advance on the position taken by the Hindus. Advanced views on imaginaries were held also by *Raphael Bombelli*, of Bologna, who published in 1572 an algebra in which he recognized that the so-called irreducible case in cubics gives real values for the roots.

It may be instructive to give examples of the algebraic notation adopted in Italy in those days.[4]

[1] See CRELLE'S *Journal*, I., 1826.

[2] Wantzel's proof, translated from Serret's *Cours d'Algèbre Supérieure*, was published in Vol. IV., p. 65, of the *Analyst*, edited by JOEL E. HENDRICKS of Des Moines. While the quintic cannot be solved by radicals, a transcendental solution, involving elliptic integrals, was given by Hermite (in the *Compt. Rend.*, 1858, 1865, 1866) and by Kronecker in 1858. A translation of the solution by elliptic integrals, taken from Briot and Bouquet's Theory of Elliptic Functions, is likewise published in the *Analyst*, Vol. V., p. 161.

[3] CANTOR, II., 508.

[4] CANTOR, II., 320; MATTHIESSEN, p. 368; the value of x given on the following page is the solution of the cubic in the previous line. The "V" or "v" is a sign of aggregation or joint root.

Pacioli : $\mathrm{R}\!\!\!/\, V\, 40\, \overline{m}\, \mathrm{R}\!\!\!/\, 320$, $\qquad \sqrt{40 - \sqrt{320}}$.

Cardan : Cubus $\bar{p}$ 6 rebus æqualis 20, $\quad x^3 + 6x = 20$,

$\mathrm{R}\!\!\!/.\ v.\ \mathrm{cu}.\ \mathrm{R}\!\!\!/.\ 108\ \bar{p}.\ 10 \mid \overline{m}\ \mathrm{R}\!\!\!/.\ v.\ \mathrm{cu}.\ \mathrm{R}\!\!\!/.\ 108\ \overline{m}\ 10$,

$$x = \sqrt[3]{\sqrt{108} + 10} - \sqrt[3]{\sqrt{108} - 10}.$$

The Italians were in the habit of calling the unknown quantity *cosa*, "thing." In Germany this word was adopted as early as the time of John Widmann as a name for algebra: he speaks of the "Regel Algobre oder Cosse"; in England this new name for algebra, the *cossic* art, gave to the first English work thereon, by Robert Recorde, its punning title the *Whetstone of Witte*, truly a *cos ingenii*. The Germans made important contributions to algebraic notation. The + and − signs, mentioned by us in the history of arithmetic, were, of course, introduced into algebra, but they did not pass into general use before the time of Vieta. "It is very singular," says Hallam, "that discoveries of the greatest convenience, and, apparently, not above the ingenuity of a village schoolmaster, should have been overlooked by men of extraordinary acuteness like Tartaglia, Cardan, and Ferrari; and hardly less so that, by dint of that acuteness, they dispensed with the aid of these contrivances in which we suppose that so much of the utility of algebraic expression consists." Another important symbol introduced by the Germans is the radical sign. In a manuscript published some time in the fifteenth century, a dot placed before a number is made to signify the extraction of a root of that number. *Christoff Rudolff*, who wrote the earliest text-book on algebra in the German language (printed 1525), remarks that "the radix quadrata is, for brevity, designated in his algorithm with the character $\surd$, as $\surd 4$." Here the dot found in the manuscript has grown into a symbol much like our own. With him VVV and VV stand for the cube and

the fourth roots. The symbol $\surd$ was used by *Michael Stifel* (1486 ?–1567), who, in 1553, brought out a second edition of Rudolff's *Coss*, containing rules for solving cubic equations, derived from the writings of Cardan. Stifel ranks as the greatest German algebraist of the sixteenth century. He was educated in a monastery at Esslingen, his native place, and afterwards became a Protestant minister. Study of the significance of mystic numbers in Revelation and in Daniel drew him to mathematics. He studied German and Italian works, and in 1544 published a Latin treatise, the *Arithmetica Integra*, given to arithmetic and algebra. Therein he observes the advantage in letting a geometric series correspond to an arithmetic series, remarking that it is possible to elaborate a whole book on the wonderful properties of numbers depending on this relation. He here makes a close approach to the idea of a *logarithm*. He gives the binomial coefficients arising in the expansion of $(a + b)$ to powers below the 18th, and uses these coefficients in the extraction of roots. German notations are illustrated by the following:

Regiomontanus: 16 census et 2000 æquales 680 rebus,

$$16\,x^2 + 2000 = 680\,x.$$

Stifel: $\surd\frac{2}{3} \cdot \surd\frac{2}{3}\, 20 - 4 - \surd\frac{5}{3}\, 8$, $\qquad \sqrt{\sqrt{20} - 4 - \sqrt[5]{8}}$.

The greatest French algebraist of the sixteenth century was *Franciscus Vieta* (1540–1603). He was a native of Poitou and died in Paris. He was educated a lawyer; his manhood he spent in public service under Henry III. and Henry IV. To him mathematics was a relaxation. Like Napier, he does not profess to be a mathematician. During the war against Spain, he rendered service to Henry IV. by deciphering intercepted letters written in a cipher of more than 500 characters with variable signification, and addressed by the Spanish court to

their governor of Netherlands. The Spaniards attributed the discovery of the key to magic. Vieta is said to have printed all his works at his own expense, and to have distributed them among his friends as presents. His *In Artem Analyticam Isagoge*, Tours, 1591, is the earliest work giving a symbolical treatment of algebra. Not only did he improve the algebra and trigonometry of his time, but he applied algebra to geometry to a larger extent, and in a more systematic manner, than had been done before him. He gave also the trigonometric solution of Cardan's irreducible case in cubics.

In the solution of equations Vieta persistently employed the principle of *reduction* and thereby introduced a uniformity of treatment uncommon in his day. He reduces affected quadratics to pure quadratics by making a suitable substitution for the removal of the term containing x. Similarly for cubics and biquadratics. Vieta arrived at a partial knowledge of the relations existing between the coefficients and the roots of an equation. Unfortunately he rejected all except positive roots, and could not, therefore, fully perceive the relations in question. His nearest approach to complete recognition of the facts is contained in the statement that the equation $x^3-(u+v+w)x^2+(uv+vw+wu)x-uvw=0$ has the three roots u, v, w. For cubics, this statement is perfect, if u, v, w, are allowed to represent any numbers. But Vieta is in the habit of assigning to letters only positive values, so that the passage really means less than at first sight it appears to do.[1] As early as 1558 *Jacques Peletier* (1517–1582), a French text-book writer on algebra and geometry, observed that the root of an equation is a divisor of the last term. Broader views were held by *Albert Girard* (1590–1633), a noted Flemish mathematician, who in 1629 issued his *Invention nouvelle en l'algèbre*.

[1] HANKEL, p. 379.

He was the first to understand the use of negative roots in the solution of geometric problems. He spoke of imaginary quantities, and inferred by induction that every equation has as many roots as there are units in the number expressing its degree. He first showed how to express the sums of their products in terms of the coefficients. The sum of the roots, giving the coefficient of the second term with the sign changed, he called the *première faction;* the sum of the products of the roots, two and two, giving the coefficient of the third term, he called *deuxième faction*, etc. In case of the equation $x^4-4x+3=0$, he gives the roots $x_1=1$, $x_2=1$, $x_3=-1+\sqrt{-2}$, $x_4=-1-\sqrt{-2}$, and then states that the imaginary roots are serviceable in showing the generality of the law of formation of the coefficients from the roots.[1] Similar researches on the theory of equations were made in England independently by *Thomas Harriot* (1560–1621). His posthumous work, the *Artis Analyticæ Praxis*, 1631, was written long before Girard's *Invention*, though published after it. Harriot discovered the relations between the roots and the coefficients of an equation in its simplest form. This discovery was therefore made about the same time by Harriot in England, and by Vieta and Girard on the continent. Harriot was the first to decompose equations into their simple factors, but as he failed to recognize imaginary, or even negative roots, he failed to prove that every equation could be thus decomposed.

Harriot was the earliest algebraist of England. After graduating from Oxford, he resided with Sir Walter Raleigh as his mathematical tutor.[2] Raleigh sent him to Virginia as surveyor in 1585 with Sir Richard Grenville's expedition. After his return, the following year, he published "A Brief and True Report of the New-found Land of Virginia," which

[1] Cantor, II., 788. [2] *Dictionary of National Biography.*

excited much notice and was translated into Latin. Among the mathematical instruments by which the wonder of the Indians was aroused, Harriot mentions "a perspective glass whereby was showed many strange sights."[1] About this time, Henry, Earl of Northumberland, became interested in Harriot. Admiring his affability and learning, he allowed Harriot a life-pension of £300 a year. In 1606 the Earl was committed to the Tower, but his three mathematical friends, Harriot, Walter Warner, and Thomas Hughes, "the three magi of the Earl of Northumberland," frequently met there, and the Earl kept a handsome table for them. Harriot was in poor health, which explains, perhaps, his failure to complete and publish his discoveries.

We next summarize the views regarding the negative and imaginary, held by writers of the sixteenth century and the early part of the seventeenth. Cardan's "pure minus" and his views on imaginaries were in advance of his age. Until the beginning of the seventeenth century mathematicians dealt exclusively with positive quantities. Pacioli says that "minus times minus gives plus," but applies this only to the formation of the product of $(a-b)(c-d)$. Purely negative quantities do not appear in his work. The German "Cossist," Rudolff, knows only positive numbers and positive roots, notwithstanding his use of the signs $+$ and $-$. His successor, Stifel, speaks of negative numbers as "less than nothing," also as "absurd numbers," which arise when real numbers above zero are subtracted from zero.[2] Harriot is the first who occasionally places a negative term by itself on one side of an equation. Vieta knows only positive numbers,

[1] Harriot was an astronomer as well as mathematician, and he "applied the telescope to celestial purposes almost simultaneously with Galileo." His telescope magnified up to 50 times. See the *Dic. of Nat. Biography*.

[2] Cantor, II., 442.

but Girard had advanced views, both on negatives and imaginaries. Before the seventeenth century, the majority of the great European algebraists had not quite risen to the views taught by the Hindus. Only a few can be said, like the Hindus, to have seen negative roots; perhaps all Europeans, like the Hindus, did not approve of the negative. The full interpretation and construction of negative quantities and the systematic use of them begins with *René Descartes* (1596–1650), but after him erroneous views respecting them appear again and again. In fact, not until the middle of the nineteenth century was the subject of negative numbers properly explained in school algebras. The question naturally arises, why was the generalization of the concept of number, so as to include the negative, such a difficult step? The answer would seem to be this: Negative numbers appeared "absurd" or "fictitious," so long as mathematicians had not hit upon a *visual or graphical representation of them.* The Hindus early saw in "opposition of direction" on a line an interpretation of positive and negative numbers. The ideas of "assets" and "debts" offered to them another explanation of their nature. In Europe full possession of these ideas was not acquired before the time of Girard and Descartes. To Stifel is due the absurd expression, negative numbers are "less than nothing." It took about 300 years to eliminate this senseless phrase from mathematical language.

History emphasizes the importance of giving graphical representations of negative numbers in teaching algebra. Omit all illustration by lines, or by the thermometer, and negative numbers will be as absurd to modern students as they were to the early algebraists.

In the development of symbolic algebra great services were rendered by Vieta. Epoch-making was the practice introduced by him of denoting general or indefinite expressions

by letters of the alphabet. To be sure, Stifel, Cardan, and others used letters before him, but Vieta first made them an essential part of algebra. The new symbolic algebra was called by him *logistica speciosa* in opposition to the old *logistica numerosa*. By his notation $a^3+3a^2b+3ab^2+b^3=(a+b)^3$ was written "a cubus $+$ b in a quadr. $3+a$ in b quadr. $3+b$ cubo æqualia $\overline{a+b}$ cubo." The vinculum was introduced by him as a sign of aggregation. Parentheses first occur with Girard. In *numerical* equations the unknown quantity was denoted by N, its square by Q, and its cube by C. Illustrations:[1]

Vieta:	$1\,C-8\,Q+16\,N$ æqu. 40,	$x^3-8x^2+16x=40.$
Vieta:	A cubus $+$ B plano 3 in A, æquari Z solido 2,	$x^3+3bx=2c.$
Girard:	$1\,③\,\infty\ 13\,①+12,$	$x^3=13x+12.$
Descartes:	$x^{3*}+px+q\infty 0,$	$x^3+px+q=0.$

Our sign of equality, $=$, is due to Recorde. Harriot adopted small letters of the alphabet in place of the capitals used by Vieta. Harriot writes $a^3-3ab^2=2c^3$ thus: $aaa-3\,bba=2\,ccc$. The symbols of inequality, $>$ and $<$, were introduced by him. William Oughtred (1574–1660) introduced $\times$ as a symbol of multiplication, : : for proportion, $\sim$ for difference. In his *Clavis*, 1631, he writes A^{10} thus, *Aqqcc*; and $120\,A^7E^3$ thus, 120 *AqqcEc*. The sign : for ratio was used by Vincent Wing in 1651; the symbol $\propto$ for variation, by William Emerson in 1768.

The Last Three Centuries

The first steps toward the building up of our modern theory of exponents and our exponential notation were taken by *Simon Stevin* (1548–1620) of Bruges in Belgium. Oresme's previous efforts in this direction remained wholly unnoticed,

[1] Matthiessen, pp. 270, 371.

but Stevin's innovations, though neglected at first, are a permanent possession. His exponential notation grew in connection with his notation for decimal fractions. Denoting the unknown quantity by ◯, he places within the circle the exponent of the power. Thus ①, ②, ③ signify x, x^2, x^3. He extends his notation to fractional exponents. ½, ⅓, ¾, mean $x^{\frac{1}{2}}$, $x^{\frac{1}{3}}$, $x^{\frac{2}{3}}$. He writes $3xyz^2$ thus $3\,①\,M\,sec\,①\,M\,ter\,②$, where M means multiplication; *sec*, second; *ter*, third unknown quantity. The ◯ for x was adopted by Girard. Stevin's great independence of mind is exhibited in his condemnation of such terms as "sursolid," or numbers that are "absurd," "irrational," "irregular," "surd." He shows that all numbers are equally proper expressions of some length, or some power of the same root. He also rejects all compound expressions, such as "square-squared," "cube-squared," and suggests that they be named by their exponents the "fourth," "fifth" powers. Stevin's symbol for the unknown failed to be adopted, but the principle of his exponential notation has survived. The modern formalism took its shape with Descartes. In his *Geometry*, 1637, he uses the last letters of the alphabet, x in the first place, then the letters y, z to designate unknown quantities; while the first letters of the alphabet are made to stand for known quantities. Our exponential notation, a^4, is found in Descartes; however, he does not use general exponents, like a^n, nor negative and fractional ones. In this last respect he did not rise to the ideas of Stevin. In case of radicals he does not indicate the root by indices, but in case of cube root, for instance, uses the letter C, thus, $\sqrt{C + \frac{1}{2}q} = \sqrt[3]{\frac{1}{2}q}$.[1]

Of the early notations for evolution, two have come down to our time, the German radical sign and Stevin's fractional

[1] Cantor, II, 794.

exponents. Modern pupils have to learn the algorithm for both notations; they must learn the meaning of $\sqrt[3]{a^2}$, and also of its equal $a^{\frac{2}{3}}$. It is a great pity that this should be so. The operations with fractional exponents are not always found easy, and the rules for radicals are always pronounced "hard." By learning both, the progress of the pupil is unnecessarily retarded. Of the two, the exponential notation is immeasurably superior. Radicals appear only in evolution.[1] Exponents, on the other hand, apply to both involution and evolution; with them all operations and simplifications are effected with comparative ease. In case of radicals, what a gain it would be, if we could burst the chains which tie us to the past!

Descartes enriched the theory of equations with a theorem which goes by the name of "rule of signs." By it are determined the number of positive and negative roots of an equation: the equation may have as many + roots as there are variations in sign, and as many − roots as there are permanences in sign. Descartes was accused by Wallis of availing himself, without acknowledgment, of Harriot's theory of equations, particularly his mode of generating equations; but there seems to be no good ground for the charge. Wallis claimed, moreover, that Descartes failed to notice that the rule breaks down in case of imaginary roots, but Descartes does not say that the equation *always has*, but that it *may have*, so many roots. It is true that Descartes does not consider the case of imaginaries directly; but further on, in his *Geometry*, he gives ample evidence of his ability to handle the case of imaginaries.

[1] In connection with the imaginary, $\sqrt{-1}$, the radical notation is objectionable, because it leads students and even authors to remark that the rules of operation for real quantities do not always hold for imaginaries, since $\sqrt{-1} \cdot \sqrt{-1}$ does not equal $\sqrt{+1}$. That the difficulty is merely one of notation is evident from the fact that it disappears when we designate the imaginary unit by i. Then $i \cdot i = i^2$, which, by definition, equals -1.

John Wallis (1616–1703) was an English mathematician of great originality. He was educated for the Church, at Cambridge, and took Holy Orders, but in 1649 was appointed Savilian professor of geometry at Oxford. He advanced beyond Kepler by making more extended use of the "law of continuity," applying it to algebra, while Desargues applied it to geometry. By this law Wallis was led to regard the denominators of fractions as powers with negative exponents. For the descending geometrical progression x^2, x^1, x^0, if continued, gives x^{-1}, x^{-2}, etc.; which is the same thing as $\frac{1}{x}$, $\frac{1}{x^2}$, etc. The exponents of his geometric series are in the arithmetical progression 2, 1, 0, -1, -2. He also used fractional exponents, which had been invented long before, but had failed to be generally introduced. The symbol ∞ for infinity is due to him. In 1685 Wallis published an *Algebra* which has long been a standard work of reference. It treats of the history, theory, and practice of arithmetic and algebra. The historical part is unreliable and worthless, but in other respects the work is a masterpiece, and wonderfully rich in material.

The study of some results obtained by Wallis on the quadrature of curves led Newton to the discovery of the Binomial Theorem, made about 1665, and explained in a letter written by Newton to Oldenburg on June 13, 1676.[1] Newton's reasoning gives the development of $(a+b)^n$, whether n be positive or negative, integral or fractional. Except when n is a positive integer, the resulting series is infinite. He gave no regular proof of his theorem, but verified it by actual multiplication. The case of positive integral exponents was proved by James Bernoulli[2] (1654–1705), by the doctrine of combinations. A

[1] How the Binomial Theorem was deduced as a corollary of Wallis's results is explained in Cajori's *History of Mathematics*, pp. 195, 196.

[2] *Ars Conjectandi*, 1713, p. 89.

proof for negative and fractional exponents was given by Leonhard Euler (1707–1783). It is faulty, for the reason that he fails to consider the convergence of the series; nevertheless it has been reproduced in elementary text-books even of recent years.[1] A rigorous general proof of the Binomial Theorem, embracing the case of incommensurable and imaginary powers, was given by Niels Henrik Abel.[2] It thus appears that for over a century and a half this fundamental theorem went without adequate proof.[3]

Sir Isaac Newton (1642–1727) is probably the greatest mathematical mind of all times. Some idea of his strong intuitive powers may be obtained from the fact that as a youth he regarded the theorems of ancient geometry as self-evident truths, and that, without any preliminary study, he made himself master of Descartes' *Geometry*. He afterwards regarded this neglect of elementary geometry as a mistake, and once expressed his regret that "he had applied himself to the works of Descartes and other algebraic writers before he had considered the *Elements* of Euclid with the attention that so excellent a writer deserves." During the first nine years of

[1] For the history of Infinite Series see Reiff, *Geschichte der Unendlichen Reihen*, Tübingen, 1889; Cantor, III., 53–94; Cajori, pp. 334–339; *Teach. and Hist. of Math. in the U.S.*, pp. 361–376.

[2] See Crelle, I., 1827, or *Œuvres complètes*, de N. H. Abel, Christiania, 1839, I., 66 *et seq.*

[3] It should be mentioned that the beginnings of the Binomial Theorem for positive integral exponents are found very early. The Hindus and Arabs used the expansions of $(a + b)^2$ and $(a + b)^3$ in the extraction of square and cube root. Vieta knew the expansion of $(a + b)^4$. But these were obtained by actual multiplication, not by any law of expansion. Stifel gave the coefficients for the first 18 powers; Pascal did similarly in his "arithmetical triangle" (see Cantor, II., 685, 686). Pacioli, Stevin, Briggs, and others also possessed something, from which one would think the Binomial Theorem could have been derived with a little attention, "if we did not know that such simple relations were difficult to discover" (De Morgan).

his professorship at Cambridge he delivered lectures on algebra. Over thirty years after, they were published, in 1707, by Mr. Whiston under the title, *Arithmetica Universalis.* They contain new and important results on the theory of equations. His theorem on the sums of powers of roots is well known. We give a specimen of his notation:

$$a^3 + 2\ aac - aab - 3\ abc + bbc.$$

Elsewhere in his works he introduced the system of literal indices. The *Arithmetica Universalis* contains also a large number of problems. Here is one (No. 50): "A stone falling down into a well, from the sound of the stone striking the bottom, to determine the depth of the well." He leaves his problems with the remark which shows that methods of teaching secured some degree of attention at his hands: "Hitherto I have been solving several problems. For in learning the sciences examples are of more use than precepts."[1]

[1] Newton's body was interred in Westminster Abbey, where in 1731 a fine monument was erected to his memory. In cyclopædias, the statement is frequently made that the Binomial Formula was engraved upon Newton's tomb, but this is very probably not correct, for the following reasons: (1) We have the testimony of Dr. Bradley, the Dean of Westminster, and of mathematical acquaintances who have visited the Abbey and mounted the monument, that the theorem can not be seen on the tomb at the present time. Yet all Latin inscriptions are still distinctly readable. (2) None of the biographers of Newton and none of the old guide-books to Westminster Abbey mention the Binomial Formula in their (often very full) descriptions of Newton's tomb. However, some of them say, that on a small scroll held by two winged youths in front of the half-recumbent figure of Newton, there are some mathematical figures. See Neale's *Guide.* Brewster, in his *Life of Sir Isaac Newton*, 1831, says that a "converging series" is there, but this does not show now. Brewster would surely have said "Binomial Theorem" instead of "converging series" had the theorem been there. The Binomial Formula, moreover, is not always convergent. (3) It is important to remember that whatever was engraved on the scroll could not be seen and read by any one, unless he stood on a chair or

The principal investigators on the solution of numerical equations are Vieta, Newton, Lagrange, Joseph Fourier, and Horner. Before Vieta, Cardan applied the Hindu rule of "false position" to cubics, but his method was crude. Vieta gave a process which is laborious, but is the forerunner of the easy methods of Newton and Horner.[1] The later changes are in the arrangement of the work, so as to afford facility and security in the evolution of the root. Horner's process is the one usually taught. *William George Horner* (1786–1837), of Bath, the son of a Wesleyan minister, was educated at Kingswood School, near Bristol. His method was read before the Royal Society, July 1, 1819.[2] Almost precisely the same method was given by the Italian Paolo Ruffini in 1804. Neither Ruffini nor Horner knew that their method had been given by the Chinese as early as the thirteenth century. De Morgan admired Horner's method and perfected it in details still further. It was his conviction that it should be included in the arithmetics; he taught it to his classes, and derided the examiners at Cambridge who ignored the method.[3] De Morgan encouraged students to carry out

used a step-ladder. Whatever was written on the scroll was, therefore, not noticed by transient visitors. The persons most likely to examine everything carefully would be the writers of the guide-books and the biographers — the very ones who are silent on the Binomial Theorem. On the other hand, such a writer as E. Stone, the compiler of the *New Mathematical Dictionary*, London, 1743, is more likely to base his statement that the theorem is "put upon the monument," upon hearsay. (4) We have the positive testimony of a most accurate writer, Augustus De Morgan, that the theorem is *not* inscribed. Unfortunately we have nowhere seen his reasons for the statement. See his article "Newton" in the *Penny* or the *English Cyclopædia*. See also our article in the *Bull. of the Am. Math. Soc.*, I., 1894, pp. 52–54.

[1] *Colo. College Publication*, General Series 51 and 52, 1910.

[2] *Dictionary of National Biography*.

[3] See De Morgan, *Budget of Paradoxes*, 2d Ed., 1915, Vol. II., pp. 188, 189.

long arithmetical computations for the sake of acquiring skill and rapidity. Thus, one of his pupils solved $x^3 - 2x = 5$ to 103 decimal places, "another tried 150 places, but broke down at the 76th, which was wrong."[1] While, in our opinion, De Morgan greatly overestimated the value of Horner's method to the ordinary boy, and, perhaps, overdid in matters of calculation, it is certainly true that in America teachers have gone to the other extreme, neglecting the *art* of rapid computation, so that our school children have been conspicuously wanting in the power of *rapid* and *accurate* figuring.[2]

In this connection we consider the approximations to the value of π. The early European computors followed the

[1] GRAVES, *Life of Sir Wm. Rowan Hamilton*, III., p. 275.

[2] The following, quoted by Mr. E. M. Langley in the *Eighteenth General Report of the A. I. G. T.*, 1892, p. 40, from De Morgan's article "On Arithmetical Computation" in the *Companion to the British Almanac* for 1844, is interesting: "The growth of the power of computation on the Continent, though considerable, did not keep pace with that of the same in England. We might give many instances of the truth of our assertion. In 1696 De Lagny, a well-known writer on algebra, and a member of the Academy of Sciences, said that the most skilful computor could not, in less than a month, find within a unit the cube root of 69653648331864003507364103 7. We would have given something to have been present if De Lagny had ever made the assertion to his contemporary, Abraham Sharp. In the present day, however, both in our universities at home and everywhere abroad, no disposition to encourage computation exists among those who attend to the higher branches of mathematics, and the elementary works are very deficient in numerical examples." De Lagny's example was brought to De Morgan in a class, and he found the root to five decimals *in less than twenty minutes.* Mr. Langley exhibits De Morgan's computation on p. 41 of the article cited. Mr. Langley and Mr. R. B. Hayward advocate Horner's method as a desirable substitute for "the clumsy rules for evolution which the young student still usually encounters in the text-books." See HAYWARD'S article in the *A. I. G. T. Report*, 1889, pp. 59–68, also DE MORGAN'S article, "Involution," in the *Penny* or the *English Cyclopædia.*

geometrical method of Archimedes by inscribed or circumscribed polygons. Thus Vieta, about 1580, computed to ten places, Adrianus Romanus (1561–1615), of Louvain, to 15 places, Ludolph van Ceulen (1540–1610) to 35 places. The latter spent years in this computation, and his performance was considered so extraordinary that the numbers were cut on his tombstone in St. Peter's churchyard at Leyden. The tombstone is lost, but a description of it is extant. After him, the value of π is often called "Ludolph's number." In the seventeenth century it was perceived that the computations could be greatly simplified by the use of infinite series. Such a series, viz. $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$, was first suggested by James Gregory in 1671. Perhaps the easiest are the formulæ used by Machin and Dase. Machin's formula is,

$$\frac{\pi}{4} = 4\tan^{-1}\frac{1}{5} - \tan^{-1}\frac{1}{239}.$$

The Englishman, Abraham Sharp, a skilful mechanic and computor, for a time assistant to the astronomer Flamsteed, took the arc in Gregory's formula equal to 30°, and calculated π to 72 places in 1705; next year Machin, professor of astronomy in London, gave π to 100 places; the Frenchman, De Lagny, about 1719, gave 127 places; the German, Georg Vesa, in 1793, 140 places; the English, Rutherford, in 1841, 208 places (152 correct); the German, Zacharias Dase, in 1844, 205 places; the German, Th. Clausen, in 1847, 250 places; the English, Rutherford, in 1853, 440 places; William Shanks, in 1873, 707 places.[1] It may be remarked that these long computations are of no theoretical or practical value. Infinitely more interesting and useful are Lambert's proof of 1761

[1] W. W. R. Ball, *Math. Recreations and Problems*, pp. 171–173. Ball gives bibliographical references.

that π is not rational,[1] and Lindemann's proof that π is not algebraical, *i.e.* cannot be the root of an algebraic equation.

Infinite series by which π may be computed were given also by Hutton and Euler. *Leonhard Euler* (1707–1783), of Basel, contributed vastly to the progress of higher mathematics, but his influence reached down to elementary subjects. He treated trigonometry as a branch of analysis, introduced (simultaneously with Thomas Simpson in England) the now current abbreviations for trigonometric functions, and simplified formulæ by the simple expedient of designating the angles of a triangle by *A*, *B*, *C*, and the opposite sides by *a*, *b*, *c*. In his old age, after he had become blind, he dictated to his servant his *Anleitung zur Algebra*, 1770, which, though purely elementary, is meritorious as one of the earliest attempts to put the fundamental processes upon a sound basis. An *Introduction to the Elements of Algebra, . . . selected from the Algebra of Euler*, was brought out in 1818 by John Farrar of Harvard College.

A question that became prominent toward the close of the eighteenth century was the graphical representation and interpretation of the imaginary, $\sqrt{-1}$. As with negative numbers, so with imaginaries, no decided progress was made until a picture of it was presented to the eye. In the time of Newton, Descartes, and Euler, the imaginary was still an algebraic fiction. The first thoroughly successful graphic representations were described by the Norwegian surveyor, Caspar Wessel, in an article of 1797, published in 1799 in the Memoirs of the Danish R. Academy of Sciences, and by *Jean Robert Argand* (1768–?), of Geneva, who in 1806 published a remarkable *Essai*.[2] But

[1] See the proof in Note IV. of Legendre's *Géométrie*, where it is extended to π^2.

[2] Consult *Imaginary Quantities. Their Geometrical Interpretation.* Translated from the French of M. Argand by A. S. Hardy, New York, 1881.

all these writings were little noticed, and it remained for the great *Carl Friedrich Gauss* (1777–1855), of Göttingen, to break down the last opposition to the imaginary. He introduced it as an independent unit co-ordinate to 1, and $a + ib$ as a "complex number." Notwithstanding the acceptance of imaginaries as "numbers" by all great investigators of the nineteenth century, there are still text-books which represent the obsolete view that $\sqrt{-1}$ is not a number or is not a quantity.

Clear ideas on the fundamental principles of algebra were not secured before the nineteenth century. As late as the latter part of the eighteenth century we find at Cambridge, England, opposition to the use of the negative.[1] The view was held that there exists no distinction between arithmetic and algebra. In fact, such writers as Maclaurin, Saunderson, Thomas Simpson, Hutton, Bonnycastle, Bridge, began their treatises with arithmetical algebra, but gradually and disguisedly introduced negative quantities. Early American writers imitated the English. But in the nineteenth century the first principles of algebra came to be carefully investigated by George Peacock,[2] D. F. Gregory,[3] De Morgan.[4] Of continental writers we may mention Augustin Louis Cauchy (1789–1857),[5] Martin Ohm,[6] and especially Hermann Hankel.[7]

[1] See C. Wordsworth, *Scholæ Academicæ: Some Account of the Studies at English Universities in the Eighteenth Century*, 1877, p. 68; *Teach. and Hist. of Math. in the U. S.*, pp. 385–387.

[2] See his *Algebra*, 1830 and 1842, and his "Report on Recent Progress in Analysis," printed in the *Reports of the British Association*, 1833.

[3] "On the Real Nature of Symbolical Algebra," *Trans. Roy. Soc. Edinburgh*, Vol. XIV., 1840, p. 280.

[4] "On the Foundation of Algebra," *Cambridge Phil. Trans.*, VII., 1841, 1842; VIII., 1844, 1847.

[5] *Analyse Algébrique*, 1821, p. 173 *et seq.*

[6] *Versuchs eines vollkommen consequenten Systems der Mathematik*, 1822, 2d Ed. 1828.

[7] *Die Complexen Zahlen*, Leipzig, 1867. This work is very rich in historical notes. Most of the bibliographical references on this subject given here are taken from that work.

A flood of additional light has been thrown on this subject by the epoch-making researches of William Rowan Hamilton, Hermann Grassmann, and Benjamin Peirce, who conceived new algebras with laws differing from the laws of ordinary algebra.[1]

GEOMETRY AND TRIGONOMETRY

Editions of Euclid. Early Researches

With the close of the fifteenth century and beginning of the sixteenth we enter upon a new era. Great progress was made in arithmetic, algebra, and trigonometry, but less prominent were the advances in geometry. Through the study of Greek manuscripts which, after the fall of Constantinople in 1453, came into possession of Western Europe, improved translations of Euclid were secured. At the beginning of this period, printing was invented; books became cheap and plentiful. The first printed edition of Euclid was published in Venice, 1482. This was the translation from the Arabic by Campanus. Other editions of this appeared, at Ulm in 1486, at Basel in 1491. The first Latin edition, translated from the original Greek, by *Bartholomæus Zambertus*, appeared at Venice in 1505. In it the translation of Campanus is severely criticised. This led Pacioli, in 1509, to bring out an edition, the tacit aim of which seems to have been to exonerate Campanus.[2] Another Euclid edition appeared in Paris, 1516. The first edition of Euclid printed in Greek was brought out in Basel in 1533, edited by *Simon Grynæus*. For 170 years this was the only Greek text. In 1703 *David Gregory* brought out at Oxford all the extant works of Euclid in the original.

[1] For an excellent historical sketch on Multiple Algebra, see J. W. Gibbs, in *Proceed. Am. Ass. for the Adv. of Science*, Vol. XXXV., 1886.

[2] Cantor, II., p. 312.

As a complete edition of Euclid, this stood alone until 1883, when Heiberg and Menge began the publication, in Greek and Latin, of their edition of Euclid's works. The first English translation of the *Elements* was made in 1570 from the Greek by "H. Billingsley, Citizen of London."[1] An English edition of the *Elements* and the *Data* was published in 1758 by *Robert Simson* (1687–1768), professor of mathematics at the University of Glasgow. His text was until recently the foundation of nearly all school editions. It differs considerably from the original. Simson corrected a number of errors in the Greek copies. All these errors he assumed to be due to unskilful editors, none to Euclid himself. A close English translation of the Greek text was made by *James Williamson*. The first volume appeared at Oxford in 1781; the second volume in 1788. School editions of the *Elements* usually contain the first six books, together with the eleventh and twelfth.

Returning to the time of the Renaissance, we mention a few

[1] In the *General Dictionary* by Bayle, London, 1735, it says that Billingsley "made great progress in mathematics, by the assistance of his friend, Mr. Whitehead, who, being left destitute upon the dissolution of the monasteries in the reign of Henry VIII., was received by Billingsley into his family, and maintained by him in his old age in his house at London." Billingsley was rich and was Lord Mayor of London in 1591. Like other scholars of his day, he confounded our Euclid with Euclid of Megara. The preface to the English edition was written by John Dee, a famous astrologer and mathematician. An interesting account of Dee is given in the *Dictionary of National Biography*. De Morgan thought that Dee had made the entire translation, but this is denied in the article "Billingsley" of this dictionary. At one time it was believed that Billingsley translated from an Arabic-Latin version, but G. B. Halsted succeeded in proving from a folio — once the property of Billingsley — [now in the library of Princeton College, and containing the Greek edition of 1533, together with some other editions] that Billingsley translated from the Greek, not the Latin. See "Note on the First English Euclid" in the *Am. Jour. of Mathem.*, Vol. II., 1879.

of the more interesting problems then discussed in geometry. Of a development of new geometrical *methods* of investigation we find as yet no trace. In his book, *De triangulis*, 1533, the German astronomer, *Regiomontanus*, gives the theorem (already known to Proclus) that the three perpendiculars from the vertices of a triangle meet in a point; and shows how to find from the three sides the radius of the circumscribed circle. He gives the first new maximum problem considered since the time of Apollonius and Zenodorus, viz., to find the point on the floor (or rather the locus of that point) from which a vertical 10 foot rod, whose lower end is 4 feet above the floor, seems largest (*i.e.* subtends the largest angle).[1] New is the following theorem, which brings out in bold relief a fundamental difference between the geometry on a plane and the geometry on a sphere: From the three angles of a spherical triangle may be computed the three sides, and *vice versa*. Regiomontanus discussed also star-polygons. He was probably familiar with the writings on this subject of Campanus and Bradwardine. Regiomontanus, and especially the Frenchman, *Charles de Bouvelles*, or Carolus Bovillus (1470–1533), laid the foundation for the theory of regular star-polygons.[2]

The construction of regular inscribed polygons received the

[1] Cantor, II., 283.

[2] A detailed history of star-polygons and star-polyædra is given by S. Günther, *Vermischte Untersuchungen*, pp. 1–92. Star-polygons have commanded the attention of geometers down even to recent times. Among the more prominent are Petrus Ramus, Athanasius Kircher (1602–1680), Albert Girard, John Broscius (a Pole), John Kepler, A. L. F. Meister (1724–1788), C. F. Gauss, A. F. Möbius, L. Poinsot (1777–1859), C. C. Krause. Möbius gives the following definition for the area of a polygon, useful in case the sides cross each other: *Given an arbitrarily formed plane polygon AB . . . MN; assume any point P in the plane and connect it with the vertices by straight lines; then the sum* $PAB + PBC + \ldots + PMN + PNA$ *is independent of the position of P and represents the area of the polygon.* Here $PAB = -PBA$.

attention of the great painter and architect, *Leonardo da Vinci* (1452–1519). Of his methods some are mere approximations, of no theoretical interest, though not without practical value. His inscription of a regular heptagon (of course, merely an approximation) he considered to be accurate! Similar constructions were given by the great German painter, *Albrecht Dürer*, (1471–1528). He is the first who always clearly and correctly states which of the constructions are approximations.[1] Both Leonardo da Vinci and Dürer in some cases perform a construction by using one single opening of the compasses. Pappus once set himself this limitation; Abûl Wafâ did this repeatedly; but now this method becomes famous. It was used by *Tartaglia* in 67 different constructions; it was employed also by his pupil *Giovanni Battista Benedetti* (1530–1590).[2]

It will be remembered that Greek geometers demanded that all *geometric* constructions be effected by a ruler and compasses only; other methods, which have been proposed from time to time, are to construct by the compasses only or by the ruler only,[3] or by ruler, compasses, and other additional instruments. Constructions of the last class were given by the Greeks, but were considered by them *mechanical*, not *geometric*. A peculiar feature in the theory of all these methods is that elementary

[1] Cantor, II., 462.

[2] For further details, consult Cantor II., 296–300, 450, 527, 529; S. Günther, *Nachträge*, p. 117, etc. The fullest development of this pretty method is reached in Steiner, *Die Geometrischen Constructionen, ausgeführt mittels der geraden Linie and eines festen Kreises*, Berlin, 1833; and in Poncelet, *Traité des propriétés projectives*, Paris, 1822, p. 187, etc.

[3] Problems to be solved by aid of the ruler only are given by Lambert in his *Freie Perspective*, Zürich, 1774; by Servois, *Solutions peu connues de différens problèmes de Géométrie pratique*, 1805; Brianchon, *Mémoire sur l'application de la théorie des transversales*. See also Chasles, p. 210; Cremona, *Elements of Projective Geometry*, Transl. by Leudesdorf, Oxford, 1885, pp. xii., 96–98.

geometry is unable to answer the general question, What constructions can be carried out by either one of these methods? For an answer we must resort to algebraic analysis.[1]

A construction by other instruments than merely the ruler and compasses appears in the quadrature of the circle by Leonardo da Vinci. He takes a cylinder whose height equals half its radius; its trace on a plane, resulting from one revolution, is a rectangle whose area is equal to that of the circle. Nothing could be simpler than this quadrature; only it must not be claimed that this solves the problem as the Greeks understood it. The ancients did not admit the use of a solid cylinder as an instrument of construction, and for good reasons: while with a ruler we can easily draw a line of any length, and, with an ordinary pair of compasses, any circle needed in a drawing, we can with a given cylinder effect not a single construction of practical value. No draughtsman ever thinks of using a cylinder.[2]

To Albrecht Dürer belongs the honour of having shown how the regular and the semi-regular solids can be constructed out of paper by marking off the bounding polygons, all in *one* piece, and then folding along the connected edges.[3]

Polyedra were a favourite study with *John Kepler*. In 1596, at the beginning of his extraordinary scientific career, he made a pseudo-discovery which brought him much fame. He placed the icosaedron, dodecaedron, octaedron, tetraedron, and cube, one within the other, at such distances that each polyedron was inscribed in the same sphere, about which the next outer one was circumscribed. On imagining the sun placed in the centre and the planets moving along great circles on the spheres — taking the radius of the sphere between the icosae-

[1] Klein, p. 2.

[2] For a good article on the "Squaring of the Circle," see Hermann Schubert in *Monist*, Jan., 1891. [3] Cantor, II., 466.

dron and dodecaedron equal to the radius of the earth's orbit — he found the distances between these planets to agree roughly with astronomical observations. This reminds us of Pythagorean mysticism. But maturer reflection and intercourse with Tycho Brahe and Galileo led him to investigations and results more worthy of his genius — "Kepler's laws." Kepler greatly advanced the theory of star-polyedra.[1] An innovation in the mode of geometrical proofs, which has since been widely used by European and American writers of elementary books, was introduced by the Frenchman, Franciscus Vieta. He considered the circle to be a polygon with an infinite number of sides.[2] The same view was taken by Kepler. In recent times this geometrical fiction has been generally abandoned in elementary works; a circle is not a polygon, but the *limit* of a polygon. To advanced mathematicians Vieta's idea is of great service in simplifying proofs, and by them it may be safely used.

The revival of trigonometry in Germany is chiefly due to *John Müller*, more generally called *Regiomontanus* (1436–1476). At Vienna he studied under the celebrated Georg Purbach, who began a translation, from the Greek, of the *Almagest*, which was completed by Regiomontanus. The latter also translated from the Greek works of Apollonius, Archimedes, and Heron. Instead of dividing the radius into 3438 parts, in Hindu fashion, Regiomontanus divided it into 600,000 equal parts, and then constructed a more accurate table of sines. Later he divided the radius into 10,000,000 parts. The *tangent* had been known in Europe before this to the Englishman, Bradwardine, but Regiomontanus went a step further, and calculated a table of tangents. He published a

[1] For drawings of Kepler's star-polyedra and a detailed history of the subject, see S. Günther, *Verm. Untersuchungen*, pp. 36–92.

[2] Cantor, II., 586.

treatise on trigonometry, containing solutions of plane and spherical triangles. The form which he gave to trigonometry has been retained, in its main features, to the present day. The task of computing accurate tables was continued by the successors of Regiomontanus. More refined astronomical instruments furnished observations of greater precision and necessitated the computation of more extended tables of trigonometric functions. Of the several tables calculated, that of *Georg Joachim* of Feldkirch in Tyrol, generally called *Rhœticus*, deserves special mention. In one of his sine-tables, he took the radius = 1,000,000,000,000,000 and proceeded from 10″ to 10″. He began also the construction of tables of tangents and secants. For twelve years he had in continual employment several calculators. The work was completed by his pupil, *Valentin Otho*, in 1596. A republication was made by Pitiscus in 1613. These tables are a gigantic monument of German diligence and perseverance. But Rhæticus was not a mere computor. Up to his time the trigonometric functions had been considered always with relation to the arc; he was the first to construct the right triangle and to make them depend directly upon its angles. It was from the right triangle that he took his idea of calculating the hypotenuse, *i.e.* the secant. His was the first European table of secants. Good work in trigonometry was done also by Vieta, Adrianus Romanus, Nathaniel Torporley, John Napier, Willebrord Snellius, Pothenot, and others. An important geodetic problem — given a terrestrial triangle and the angles subtended by the sides of the triangle at a point in the same plane, to find the distances of the point from the vertices — was solved by Snellius in a work of 1617, and again by Pothenot in 1730. Snellius's investigation was forgotten and it secured the name of "Pothenot's problem." The word "radian" is due to James Thomson, 1871.

The Beginning of Modern Synthetic Geometry

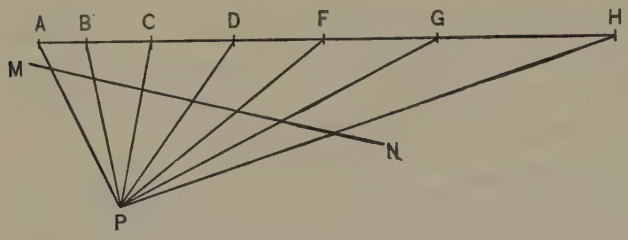
About the beginning of the seventeenth century the first decided advance, since the time of the ancient Greeks, was made in Geometry. Two lines of progress are noticeable: (1) the analytic path, marked out by the genius of Descartes, the inventor of Analytical Geometry; (2) the synthetic path, with the new principle of perspective and the theory of transversals. The early investigators in modern synthetic geometry are Desargues, Pascal, and De Lahire.

Girard Desargues (1593–1662), of Lyons, was an architect and engineer. Under Cardinal Richelieu he served in the siege of La Rochelle, in 1628. Soon after, he retired to Paris, where he made his researches in geometry. Esteemed by the ablest of his contemporaries, bitterly attacked by others unable to appreciate his genius, his works were neglected and forgotten, and his name fell into oblivion until, in the early part of the nineteenth century, it was rescued by Brianchon and Poncelet. Desargues, like Kepler and others, introduced the doctrine of infinity into geometry.[1] He states that the straight line may be regarded as a circle whose centre is at infinity; hence, the two extremities of a straight line may be considered as meeting at infinity; parallels differ from other pairs of lines only in having their points of intersection at infinity. He gives the theory of involution of six points, but his definition of "involution" is not quite the same as the modern definition, first found in Fermat,[2] but really introduced into geometry by Chasles.[3] On a line take the point A as origin (*souche*), take also the three pairs of points B and H, C and G, D and F; then, says Desargues, if $AB \cdot AH = AC \cdot AG$

[1] Charles Taylor, *Introduction to the Ancient and Modern Geometry of Conics*, Cambridge, 1881, p. lxi. [2] Cantor, II., 677, 678.

[3] Consult Chasles, *Note* X.; Marie, III., 214.

$= AD \cdot AF$, the six points are in "involution." If a point falls on the origin, then its partner must be at an infinite distance from the origin. If from any point P lines be drawn through the six points, these lines cut any transversal MN in six other points, which are also in involution; that is, *involution is a projective relation.* Desargues also gives the theory of polar lines. What is called "Desargues' Theorem" in elementary works is as follows: If the vertices of two triangles, situated either in space or in a plane, lie on three lines meeting in a



point, then their sides meet in three points lying on a line, and conversely. This theorem has been used since by Brianchon, Sturm, Gergonne, and others. Poncelet made it the basis of his beautiful theory of homological figures.

Although the papers of Desargues fell into neglect, his ideas were preserved by his disciples, Pascal and *Philippe de Lahire.* The latter, in 1679, made a complete copy of Desargues' principal research, published in 1639. *Blaise Pascal* (1623–1662) was one of the very few contemporaries who appreciated the worth of Desargues. He says in his *Essais pour les coniques,* "I wish to acknowledge that I owe the little that I have discovered on this subject to his writings." Pascal's genius for geometry showed itself when he was but twelve years old. His father wanted him to learn Latin and Greek before entering on mathematics. All mathematical books were hidden out of sight. In answer to a question, the boy was told by

his father that mathematics "was the method of making figures with exactness, and of finding out what proportions they relatively had to one another." He was at the same time forbidden to talk any more about it. But his genius could not be thus confined; meditating on the above definition, he drew figures with a piece of charcoal upon the tiles of the pavement. He gave names of his own to these figures, then formed axioms, and, in short, came to make perfect demonstrations. In this way he arrived, unaided, at the theorem that the angle-sum in a triangle is two right angles. His father caught him in the act of studying this theorem, and was so astonished at the sublimity and force of his genius as to weep for joy. The father now gave him Euclid's *Elements*, which he mastered easily. Such is the story of Pascal's early boyhood, as narrated by his devoted sister.[1] While this narrative must be taken *cum grano salis* (for it is highly absurd to suppose that young Pascal or any one else could re-discover geometry as far as Euclid I., 32, following the same treatment and hitting upon the same sequence of propositions as found in the *Elements*), it is true that Pascal's extraordinary penetration enabled him at the age of sixteen to write a treatise on conics which passed for such a surprising effort of genius that it was said nothing equal to it in power had been produced since the time of Archimedes. Descartes refused to believe that it was written by one so young as Pascal. This treatise was never published, and is now lost. Leibniz saw it in Paris, recommended its publication, and reported on a portion of its contents.[2] However, Pascal published in 1640, when he was sixteen years old, a small geometric treatise of six octavo

[1] *The Life of Mr. Paschal*, by Madam Perier. Translated into English by W. A., London, 1744.

[2] See letter written by Leibniz to Pascal's nephew, August 30, 1676, which is given in *Oeuvres complètes de Blaise Pascal*, Paris, 1866, Vol. III.,

pages, bearing the title, *Essais pour les coniques.* Constant application at a tender age greatly impaired Pascal's health. During his adult life he gave only a small part of his time to the study of mathematics.

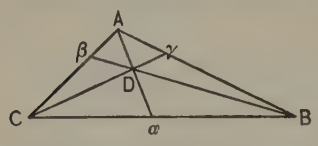
Pascal's two treatises just noted contained the celebrated proposition on the mystic hexagon, known as "Pascal's Theorem," viz. that the opposite sides of a hexagon inscribed in a conic intersect in three points which are collinear. In our elementary text-books on modern geometry this beautiful theorem is given in connection with a very special type of a conic, namely, the circle. As, in one sense, any two straight lines may be looked upon as a special case of a conic, the theorem applies to hexagons whose first, third, and fifth vertices are on one line, and whose second, fourth, and sixth vertices are on the other. It is interesting to note that this special case of "Pascal's Theorem" occurs already in Pappus (Book VII., Prop. 139). Pascal said that from his theorem he deduced over 400 corollaries, embracing the conics of Apollonius and many other results. Pascal gave the theorem on the cross ratio, first found in Pappus.[1] This wonderfully fruitful theorem may be stated as follows: Four lines in a plane, passing through one common point, cut off four segments on a transversal which have a fixed, constant ratio, in whatever manner the transversal may be drawn; that is, if the transversal cuts the rays in the points A, B, C, D, then the ratio $\frac{AC}{AD} : \frac{BC}{BD}$, formed by the four segments AC, AD, BC, BD, is the same for all transversals. The researches of Desargues and Pascal uncovered several of the rich treasures

pp. 466–468. The *Essais pour les coniques* is given in Vol. III., pp. 182–185, of the *Oeuvres complètes*, also in *Oeuvres de Pascal* (The Hague, 1779) and by H. Weissenborn in the preface to his book, *Die Projection in der Ebene*, Berlin, 1862.

[1] *Book* VII., 129. Consult Chasles, pp. 31, 32.

of modern synthetic geometry; but owing to the absorbing interest taken in the analytical geometry of Descartes, and, later, in the differential calculus, the subject was almost entirely neglected until the close of the eighteenth century.

Synthetic geometry was advanced in England by the researches of *Sir Isaac Newton*, *Roger Cotes* (1682–1716), and *Colin Maclaurin*, but their investigations do not come within the scope of this history. *Robert Simson* and *Matthew Stewart* (1717–1785) exerted themselves mainly to revive Greek geometry. An Italian geometer, *Giovanni Ceva* (1647–1734)[1] deserves mention here; a theorem in elementary geometry bears his name. He was an hydraulic engineer, and as such was several times employed by the government of Mantua. His death took place during the siege of Mantua, in 1734. He ranks as a remarkable author in economics, being the first clear-sighted mathematical writer on this subject. In 1678 he published in Milan a work, *De lineis rectis*. This contains "Ceva's Theorem" with one static and two geometric proofs:



Any three concurrent lines through the vertices of a triangle divide the opposite sides so that $C\alpha \cdot A\beta \cdot B\gamma = B\alpha \cdot C\beta \cdot A\gamma$. In Ceva's book the properties of rectilinear figures are proved by considering the properties of the centre of inertia (gravity) of a system of points.[2]

Modern Elementary Geometry

We find it convenient to consider this subject under the following four sub-heads: (1) Modern Synthetic Geometry, (2) Modern Geometry of the Triangle and Circle, (3) Non-

[1] *Bulletin Am. Math. Soc.*, Vol. 22, 1915, p. 100.
[2] Chasles, Notes VI., VII.

Euclidean Geometry, (4) Text-books on Elementary Geometry. The first of these divisions has reference to modern synthetic *methods of research*, the second division refers to *new theorems* in elementary geometry, the third considers the modern *conceptions of space* and the several geometries resulting therefrom, the fourth discusses questions pertaining to *geometrical teaching*.

I. *Modern Synthetic Geometry.* — It was reserved for the genius of *Gaspard Monge* (1746–1818) to bring modern synthetic geometry into the foreground, and to open up new avenues of progress. To avoid the long arithmetical computations in connection with plans of fortification, this gifted engineer substituted geometric methods and was thus led to the creation of descriptive geometry as a distinct branch of science. Monge was professor at the Normal School in Paris during the four months of its existence, in 1795; he then became connected with the newly established Polytechnic School, and later accompanied Napoleon on the Egyptian campaign. Among the pupils of Monge were Dupin, Servois, Brianchon, Hachette, Biot, and Poncelet. *Charles Julien Brianchon*, born in Sèvres in 1785, deduced the theorem, known by his name, from "Pascal's Theorem" by means of Desargues' properties of what are now called polars.[1] Brianchon's theorem says: "The hexagon formed by any six tangents to a conic has its opposite vertices connecting concurrently." The point of meeting is sometimes called the "Brianchon point."

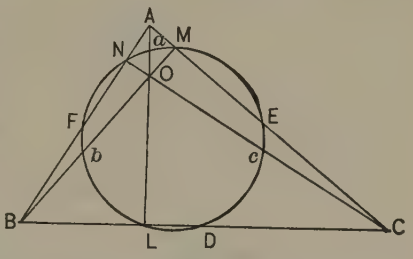
Lazare Nicholas Marguerite Carnot (1753–1823) was born at Nolay in Burgundy. At the breaking out of the Revolution he threw himself into politics, and when coalesced Europe, in 1793, launched against France a million soldiers, the gigantic

[1] Brianchon's proof appeared in "Memoirs sur les Surfaces courbes du second Degré," in *Journal de l'Ecole Polytechnique*, T. VI., 297–311, 1806. It is reproduced by Taylor, *op. cit.*, p. 290.

task of organizing fourteen armies to meet the enemy was achieved by him. He was banished in 1796 for opposing Napoleon's *coup d'état*. His *Geometry of Position*, 1803, and his *Essay on Transversals*, 1806, are important contributions to modern plane geometry. By his effort to explain the meaning of the negative sign in geometry he established a "geometry of position" which, however, is different from Von Staudt's work of the same name. He invented a class of general theorems on projective properties of figures, which have since been studied more extensively by Poncelet, Chasles, and others.

Jean Victor Poncelet (1788–1867), a native of Metz, engaged in the Russian campaign, was abandoned as dead on the bloody field of Krasnoi, and from there taken as prisoner to Saratoff. Deprived of books, and reduced to the remembrance of what he had learned at the Lyceum at Metz and the Polytechnic School, he began to study mathematics from its elements. Like Bunyan, he produced in prison a famous work, *Traité des Propriétés projectives des Figures*, first published in 1822. Here he uses central projection, and gives the theory of "reciprocal polars." To him we owe the Law of Duality as a consequence of reciprocal polars. As an independent principle it is due to *Joseph Diaz Gergonne* (1771–1859). We can here do no more than mention by name a few of the more recent investigators: *Augustus Ferdinand Möbius* (1790–1868), *Jacob Steiner* (1796–1863), *Michel Chasles* (1793–1880), *Karl Georg Christian von Staudt* (1798–1867). Chasles introduced the bad term *anharmonic ratio*, corresponding to the German *Doppelverhältniss* and to Clifford's more desirable *cross-ratio*. Von Staudt cut loose from all algebraic formulæ and from metrical relations, particularly the metrically founded cross-ratio of Steiner and Chasles, and then created a geometry of position, which is a complete science in itself, independent of all measurement.

II. *Modern Geometry of the Triangle and Circle.* — We cannot give a full history of this subject, but we hope by our remarks to interest a larger circle of American readers in the recently developed properties of the triangle and circle.[1] Frequently quoted in recent elementary geometries is the "nine-point circle." In the triangle ABC, let D, E, F be the middle points of the sides, let AL, BM, CN be perpendiculars to the sides, let a, b, c be the middle points of AO, BO, CO, then a circle can be made to pass through the points, L, D, c, E, M, a, N, F, b; this circle is the "nine-point circle." By mistake, the earliest discovery of this circle has been attributed to Euler.[2] There are several independent discoverers. In England, *Benjamin Bevan* proposed in Leybourn's *Mathematical Repository*, I., 18, 1804, a theorem for proof which practically gives us the nine-point circle.



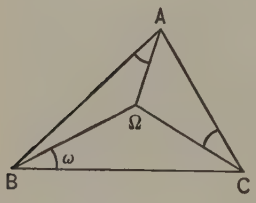
[1] A systematic treatise on this subject, which we commend to students is A. Emmerich's *Die Brocardschen Gebilde*, Berlin, 1891. Our historical notes are taken from this book and from the following papers: Julius Lange, *Geschichte des Feuerbachschen Kreises*, Berlin, 1894; J. S. Mackay. *History of the nine-point circle*, pp. 19–57, *Early history of the symmedian point*, pp. 92–104, in the *Proceed. of the Edinburgh Math. Soc.*, Vol. XI., 1892–93. See also Mackay, *The Wallace line and the Wallace point* in the same journal, Vol. IX., 1891, pp. 83–91; E. Lemoine's paper in *Association française pour l'avancement des Sciences*, Congrès de Grenoble, 1885; E. Vigarié, in the same publication, Congrès de Paris, 1889. The progress in the geometry of the triangle is traced by Vigarié for the year 1890 in *Progreso mat.* I., 101–106, 128–134, 187–190; for the year 1891 in *Journ. de Math. élém.*, (4) 1. 7–10, 34–36. Consult also Casey, *Sequel to Euclid.*

[2] Mackay, *op. cit.*, Vol. XI., p. 19.

The proof was supplied to the *Repository*, Vol. I., Part 1, p. 143, by *John Butterworth*, who also proposed a problem, solved by himself and *John Whitley*, from the general tenor of which it appears that they knew the circle in question to pass through all nine points. These nine points are explicitly mentioned by Brianchon and Poncelet in Gergonne's *Annales de Mathématiques* of 1821. In 1822, *Karl Wilhelm Feuerbach* (1800–1834) professor at the gymnasium in Erlangen, published a pamphlet in which he arrives at the nine-point circle, and proves that it touches the incircle and the ex-circles. The Germans called it "Feuerbach's Circle." Many demonstrations of its characteristic properties are given in the article above referred to. The last independent discoverer of this remarkable circle, so far as known, is F. S. Davies, in an article of 1827 in the *Philosophical Magazine*, II., 29–31.

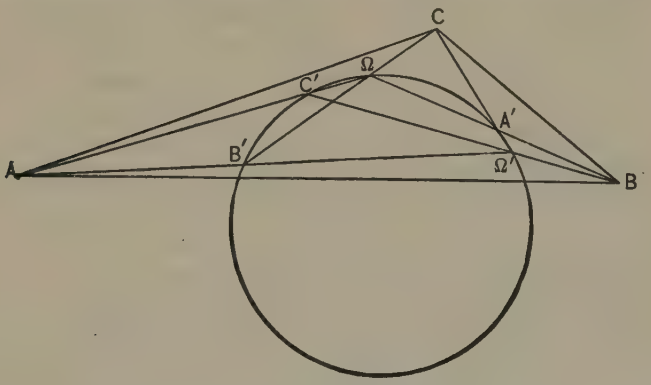
In 1816 *August Leopold Crelle* (1780–1855), the founder of a mathematical journal bearing his name, published in Berlin a paper dealing with certain properties of plane triangles. He showed how to determine a point Ω inside a triangle, so that the angles (taken in the same order) formed by the lines joining it to the vertices are equal.

In the adjoining figure the three marked angles are equal.



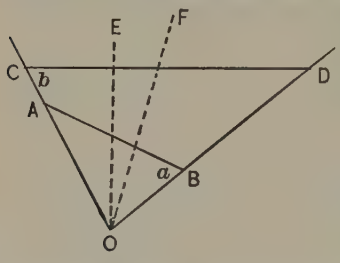
If the construction be made so as to give angle $\Omega'AC = \Omega'CB = \Omega'BA$, then a second point Ω' is obtained. The study of the properties of these new angles and new points led Crelle to exclaim: "It is indeed wonderful that so simple a figure as the triangle is so inexhaustible in properties. How many as yet unknown properties of other figures may there not be!" Investigations were made also by *C. F. A. Jacobi* of Pforta and some of his pupils, but after his death, in 1855, the whole matter

was forgotten. In 1875 the subject was again brought before the mathematical public by *H. Brocard*, who had taken up this study independently a few years earlier. The work of Brocard was soon followed up by a large number of investigators in France, England, and Germany. The new researches gave rise to an extended new vocabulary of technical terms. Unfortunately, the names of geometricians which have been attached to certain remarkable points, lines, and circles are not always the names of the men who first studied their properties. Thus, we speak of "Brocard points" and "Brocard angles," but



historical research brought out the fact, in 1884, and 1886, that these were the points and lines which had been studied by Crelle and C. F. A. Jacobi. The "Brocard Circle," is Brocard's own creation. In the triangle ABC, let Ω and Ω' be the first and second "Brocard point." Let A' be the intersection of $B\Omega$ and $C\Omega'$; B', of $A\Omega'$ and $C\Omega$; C', of $B\Omega'$ and $A\Omega$. The circle passing through A', B', C' is the "Brocard circle." $A'B'C'$ is "Brocard's first triangle." Another like triangle, $A''B''C''$ is called "Brocard's second triangle." The points A'', B'', C'' together with Ω, Ω', and two other points, lie in the circumference of the "Brocard circle."

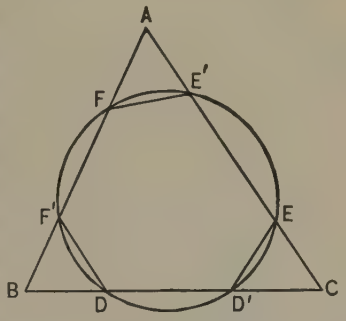
In 1873 *Emile Lemoine* called attention to a particular point within a plane triangle which since has been variously called the "Lemoine point," "symmedian point," and "Grebe point." Since that time the properties of this point and of the lines and circles connected with it have been diligently investigated. To lead up to its definition we premise that, in the adjoining figure, if CD is so drawn as to make angles a and b equal, then

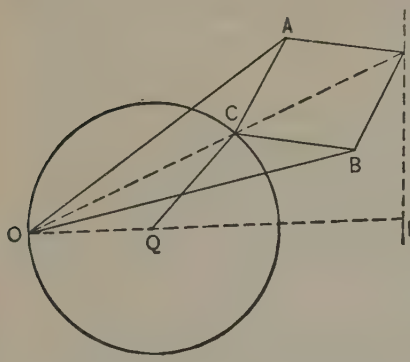


one of the two lines AB and CD is the *anti-parallel* of the other, with reference to the angle O.[1] Now OE, the bisector of AB, is the *median* and OF, the bisector of the anti-parallel of AB, is called the *symmedian* (abbreviated from *symétrique de la médiane*). The point of concurrence of the three symmedians in a triangle is called, after Tucker, the "symmedian point." Mackay has pointed out that some of the properties of this point, recently brought to light, were discovered previously to 1873. The anti-parallels of a triangle which pass through its symmedian point, meet its sides in six points which lie on a circle, called the "second Lemoine circle." The "first Lemoine circle" is a special case of a "Tucker circle" and concentric with the "Brocard circle." The "Tucker circles" may be thus defined. Let $DF' = FE' = ED'$; let, moreover, the following pairs of lines be anti-parallels to each other:

[1] The definition of anti-parallels is attributed by EMMERICH (p. 13, note) to Leibniz. The term anti-parallel is defined by E. STONE in his *New Mathem. Dict.*, London, 1743. Stone gives the above definition, refers to Leibniz in *Acta Erudit.*, 1691, p. 279, and attributes to Leibniz a definition different from the above. The word anti-parallel is given in MURRAY'S *New English Dictionary* of about 1660. See *Jahrbuch über die Fortschritte der Mathematik*, Bd. XXII., 1890, p. 45; *Nature*, XLI., 104–105.

AB and ED', BC and FE', CA and DF'; then the six points D, D', E, E', F, F', lie on a "Tucker circle." Vary the length of the equal anti-parallels, and the various "Tucker circles" are obtained. Allied to these are the "Taylor circles." Still different types are the "Neuberg circles" and "Mackay circles." Perhaps enough has been said to call to mind the wonderful advance which has been made in the geometry of the triangle and circle during the latter half of the nineteenth century. That new theorems should have been found in recent time is the more remarkable when we consider that these figures were subjected to close examination by the keen-minded Greeks and the long line of geometers who have since appeared.[1]





Of interest is the invention in 1853 by Sarrut of an apparatus for generating rectilinear motions; it has been altogether overlooked. Celebrated is the invention in 1864 of an apparatus for converting circular into rectilinear motion,[2] by A. Peaucellier, a French army officer. Let

[1] For a more detailed statement of researches in England see article "The Recent Geometry of the Triangle," *Fourteenth General Report* of A. I. G. T., 1888, pp. 35–46.

[2] PEAUCELLIER's articles appeared in *Nouvelles Annales*, 1864 and 1873

$ACBP$ be a rhombus whose sides are less than the equal sides of the angle AOB. Imagine the straight lines which are drawn in full to be bars or "links," jointed at the points A, B, C, P, Q, O. Make C describe the circle and P will describe the straight line PD, the pivot O being fixed. Remove CQ, let C move along any curve in the plane, then P will trace the inverse of that curve with respect to O. Peaucellier's method of linkages was developed further by Sylvester.[1]

Elsewhere we have spoken of the instruments used in geometrical constructions — how the Greeks used the ruler and compasses, and how later a ruler with a fixed opening of the compasses, or the ruler alone, came to be used by a few geometers. Of interest in this connection is a work by the Italian *Lorenzo Mascheroni* (1750–1800), entitled *Geometria del compasso*, 1797, in which all constructions are made with a pair of compasses, but without restriction to a fixed radius. The book was written for the practical mechanic, the author claiming that constructions with compasses are more accurate than those with a ruler.[2] The work secured the attention of Napoleon Bonaparte, who proposed to the French mathematicians the following problem: To divide the circumference of a circle into four equal parts by the compasses only. This construction is as follows: Apply the radius three times to the circumference, giving the arcs AB, BC, CD. Then the distance AD is a diameter. With a radius equal to AC, and

[1] Sylvester in *Educ. Times Reprint*, Vol. XXI., 58, (1874). See C. Taylor, *Ancient and Modern Geometry of Conics*, 1881, p. LXXXVII. Consult also A. B. Kempe, *How to draw a straight line, a lecture on linkages*, London, 1877.

[2] See Charles Hutton's *Philos. and Math. Dict.*, London, 1815, article "Geometry of the Compasses"; Marie, X. p. 98; Klein, p. 26; Steiner, *Gesammelte Werke*, I., 463; Mascheroni's book was translated into French in 1798; an abridgment of it by Hutt was brought out in German in 1880.

with centres A and D, draw arcs intersecting at E. Then EO, where O is the centre of the given circle, is the chord of the quadrant of the circle.

The inscription of the regular polygon of 17 sides was first effected by *Carl Friedrich Gauss* (1777–1855), when a boy of nineteen, at the University of Göttingen, March 30, 1796. At that time he was undecided whether to choose ancient languages or mathematics as his specialty. His success in this inscription led to the decision in favour of mathematics.[1]

A curious mode of construction has been suggested independently by a German and a Hindu. Constructions are to be effected by the *folding of paper*. *Hermann Wiener*, in 1893, showed how to construct, by folding, the nets of the regular solids. In the same year, *Sundara Row* published a little book "On paper folding" (Addison & Co., Madras), in which it is shown how to construct any number of points on the ellipse, cissoid, etc.[2]

In connection with polyedra the theorem that the number of edges falls short by two of the combined number of vertices and faces is interesting. The theorem is usually ascribed to Euler, but was worked out earlier by Descartes.[3] It is true only for polyedra whose faces have each but one boundary; if a cube is placed upon a larger cube, then the upper face of the larger cube has two boundaries, an inner and an outer, and the theorem is not true. A more general theorem is due to *F. Lippich*.[4]

[1] Other modes of inscription of the 17-sided polygon were given by Von Staudt, in *Crelle*, 24 (1842), by Schröter, in *Crelle*, 75 (1872). The latter uses a ruler and a single opening of the compasses. By the compasses only, the inscription has also been effected. See Klein, p. 27; an inscription is given also in Paul Bachmann, *Kreistheilung*, Leipzig, 1872, p. 67.

[2] Klein, p. 33.

[3] See E. de Jonquières in *Biblioth. Mathem.*, 1890, p. 43.

[4] Lippich, "zur Theorie der Polyeder" *Sitz.-Ber d. Wien. Akad.*, Bd.

III. *Non-Euclidean Geometry.* — The history of this subject centres almost wholly in the theory of parallel lines. Preliminary to the discussion of the non-Euclidean geometries, it may be profitable to consider the various efforts towards simplifying and improving the parallel-theory made, (1) by giving a new definition of parallel lines, or by assuming a new postulate, different from Euclid's parallel-postulate, (2) by proving the parallel-postulate from the nature of the straight line and plane angle.

Euclid's definition, *parallel straight lines are such as are in the same plane, and which being produced ever so far both ways do not meet,* still holds its place as the best definition for use in elementary geometry. A new definition, used by Posidonius (1st Cent. B.C.), was adopted by the painter, *Albrecht Dürer.* He wrote a geometry, first printed in 1525, in which *parallel lines are lines everywhere equally distant.*[1] A little later *Clavius,* in his edition of Euclid of 1574, in a remark, assumes that a line which is everywhere equidistant from a straight line is itself straight. This postulate lies hidden in the definition of Dürer. The objection to this definition or postulate is that it is an advanced *theorem,* involving the difficult consideration of measurement, embracing the whole theory of incommensurables. Moreover, in a more general view of geometry it must be abandoned.[2] Clavius's treatment is adopted by *Jacques Peletier* (1517–1582) of Paris, *Ruggiero Giuseppe Boscovich* (1711–1787),[3] *Johann Christian Wolf* (1679–1754)

84, 1881; see also H. Durège, *Theorie der Funktionen,* Leipzig, 1882, p. 226; an English translation of this work, made by G. E. Fisher and I. J. Schwatt, has appeared.

[1] S. Günther, *Math. Unt. im d. Mittela.*, pp. 361, 362.

[2] See Lobatchewsky, *The Theory of Parallels,* transl. by G. B. Halsted, Austin, 1891, p. 21, where the theorem is proved that in pseudo-spherical space, "The farther parallel lines are prolonged on the side of their parallelism, the more they approach one another."

[3] He was professor at several Italian universities, was employed in

of Halle, *Thomas Simpson* (in the first edition of his Elements, 1747), *John Bonnycastle* (1750 ?–1821) of the Royal Military Academy at Woolwich, and others. It has been used by a few American authors.[1] This definition is the first of several given by *E. Stone* in his *New Mathematical Dictionary*, London, 1743; in fact, "in the majority of text-books on elementary geometry, from the sixteenth to the beginning of the eighteenth century, parallel lines are declared to be lines that are equidistant — which is, to be sure, very convenient."[2] But objections to this mode of treatment soon arose. As early as 1680 *Giordano da Bitonto*, in Italy, said that it is inadmissible, unless the actual existence of equidistant straight lines can be established. *Saccheri* rejects the assumption unceremoniously.[3]

Another definition, which involves a tacit assumption, declares parallels to be *lines which make the same angle with a third line.* The definition appears to have originated in France; it is given by *Pierre Varignon* (1654–1722) and *Étienne Bézout* (1730–1783), both of Paris. In England it was used by Cooley,[4] in the United States by H. N. Robinson. A slight modification of this is the following definition: *Parallels are lines perpendicular to a third line.* This is recom-

various scientific duties by several popes, was in London in 1762, and was recommended by the Royal Society as a proper person to be appointed to observe the transit of Venus *at California*, but the suppression of the Jesuit order, which he had entered, prevented his acceptance of the appointment. See *Penny Cyclopædia.*

[1] Consult *Teach. and Hist. of Math. in the U. S.*, p. 377.

[2] Engel and Stäckel, p. 33.

[3] Engel and Stäckel, p. 46.

[4] "Minos: So far as I can make out Mr. Cooley quietly assumes that a pair of lines which make equal angles with *one* line, do so with *all* lines. He might just as well say that a young lady, who was inclined to one young man, was 'equally and similarly inclined to *all* young men'! Rhadamanthus: She might 'make equal angling' with them all any-

mended by the Italian G. A. Borelli in 1658, and by the celebrated French text-book writer, *S. F. Lacroix.*[1]

Famous is the definition, *parallel lines are straight lines having the same direction.* At first, this definition attracts us by its simplicity. But the more it is studied, the more perplexing are the questions to which it gives rise. Strangely enough, authors adopting this definition encounter no further difficulty with parallel lines; nowhere do they meet the necessity of assuming Euclid's parallel-postulate or any equivalent of it! A question which has perplexed geometers for centuries appears disposed of in a trice! In actual fact, there is, perhaps, no word in mathematics which, by its apparent simplicity, but real indefiniteness and obscurity, has misled so many able minds. In the United States, as elsewhere, this definition has been widely used, but for the sake of sound learning, it should be banished from text-books forever.[2]

A new definition of parallel lines, suggested by the principle of continuity, and one of great assistance in advanced geometry, though unsuited for elementary instruction, is that first

how." C. L. Dodgson, *Euclid and His Modern Rivals*, London, 1885, 2d Ed., p. 2; also p. 62.

[1] S. F. Lacroix, *Essais sur l'enseignement en général, et sur celui des mathématiques en particulier*, Paris, 1805, p. 317.

[2] The objections to the term "direction" have been ably set forth so often, that we need only refer to some of the articles: De Morgan's review of J. M. Wilson's Geometry, *Athenæum*, July 18, 1868; E. L. Richards in *Educational Review*, Vol. III., 1892, p. 32; *Seventh General Report* (1891) of the A. I. G. T., pp. 36–42; G. B. Halsted, *The Science Absolute of Space by* John Bolyai, 4th Ed., 1896, pp. 63–71; G. B. Halsted in *Educational Review*, Sept., 1893, p. 153; C. L. Dodgson, *Euclid and His Modern Rivals*, London; H. Müller, *Besitzt die heutige Schulgeometrie noch die Vorzüge des Euklidischen Originals?* Metz, 1889, p. 7; *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, Teubner in Leipzig, XVI., p. 407; *Teach. and Hist. of Math. in the U. S.*, pp. 381–383; *Monist*, 1892 (Review of E. T. Dixon's "Foundations of Geometry").

given by John Kepler (1604) and Girard Desargues (1639): Lines are parallel if they have the same infinitely distant point in common. A similar idea is expressed by E. Stone in his dictionary thus: "If A be a point without a given indefinite right line CD; the shortest line, as AB, that can be drawn from A to it, is perpendicular; and the longest, as EA, is parallel to CD."

Quite a number of substitutes for Euclid's parallel-postulate differing in form, but not in essence, have been suggested at various times. In the evening of July 11, 1663, *John Wallis* delivered a lecture at Oxford on the parallel-postulate.[1] He recommends in place of Euclid's postulate: *To any triangle another triangle, as large as you please, can be drawn, which is similar to the given triangle.* *Saccheri* showed that Euclidean geometry can be rigidly developed if the existence of one triangle, unequal but similar to another, be presupposed. Lambert makes similar remarks. Wallis's postulate was again proposed for adoption by L. Carnot and Laplace, and more recently by J. Delbœuf.[2] *Alexis Claude Clairaut* (1713–1765), a famous French mathematician, wrote an elementary geometry, in which he *assumes the existence of a rectangle*, and from this substitute for Euclid's postulate develops the elementary theorems with great clearness. Other equivalent postulates are the following: a circle can be passed through any three points not in the same straight line (due to W. Bolyai); the existence of a finite triangle whose angle-sum is two right angles (due to Legendre); through every point within an angle a line can be drawn intersecting both sides (due to J. F. Lorenz (1791), and Legendre); in any circle, the inscribed equilateral quadrangle is greater than any one of the segments

[1] See *Opera*, Vol. II., 665–678. In German translation it is given by Engel and Stäckel, pp. 21–30.

[2] Engel and Stäckel, p. 19.

which lie outside of it (due to C. L. Dodgson); two straight lines which intersect each other cannot be both parallel to the same straight line (used by John Playfair).[1] Of all these substitutes, only the last has met with general favour. Playfair adopted it in his edition of Euclid, and it has been generally recognized as simpler than Euclid's parallel-postulate.

Until the close of the first quarter of the nineteenth century it was widely believed by mathematicians that Euclid's parallel-postulate could be *proved* from the other assumptions and the definitions of geometry. We have already referred to such efforts made by Ptolemy and Nasîr Eddîn. We refrain from discussing proofs in detail. They all fail, either because an equivalent assumption is implicitly or explicitly made, or because the reasoning is otherwise fallacious. On this slippery ground good and bad mathematicians alike have fallen. We are told that the great Lagrange, noticing that the formulæ of spherical trigonometry are not dependent upon the parallel-postulate, hoped to frame a proof on this fact. Toward the close of his life he wrote a paper on parallel lines and began to read it before the Academy, but suddenly stopped and said: "Il faut que j'y songe encore"[2] (I must think it over again); he put the paper in his pocket and never afterwards publicly recurred to it.

The researches of *Adrien Marie Legendre* (1752–1833) are interesting. Perceiving that Euclid's postulate is equivalent to the theorem that the angle-sum of a triangle is two right angles, he gave this an analytical proof, which, however, assumes the existence of similar figures. Legendre was not satisfied with this. Later, he proved satisfactorily by assuming lines to be of indefinite extent, that the angle-sum can-

[1] Consult G. B. Halsted's *Bolyai*, 4th Ed., pp. 64, 65.
[2] Engel and Stäckel, p. 212.

not exceed two right angles, but could not prove that it cannot fall short of two right angles. In 1823, in the twelfth edition of his *Elements of Geometry*, he thought he had a proof for the second part. Afterwards, however, he perceived its weakness, for it rested on the new assumption that through any point within an angle a line can be drawn, cutting both sides of the angle. In 1833 he published his last paper on parallels, in which he correctly proves that, if there be any triangle the sum of whose angles is two right angles, then the same must be true of all triangles. But in the next step, to show rigorously the actual existence of such a triangle, his demonstration failed, though he himself thought he had finally settled the whole question.[1] As a matter of fact he had not gotten quite so far as had Saccheri one hundred years earlier.[2] Moreover, before the publication of his last paper, a Russian mathematician had taken a step which far transcends in boldness and importance anything Legendre had done on this subject.

As with the problem of the quadrature of the circle, so with the parallel-postulate: after numberless failures on the part of some of the best minds to resolve the difficulty, certain shrewd thinkers began to suspect that the postulate

[1] A novel attempt to prove the angle-sum of a triangle to be two right angles was made in 1813 by John Playfair, in his *Elements of Geometry*. See the edition of 1855, Philadelphia, pp. 295, 296. It consists, briefly, in laying a straight edge along one of the sides of the figure, and then turning the edge round so as to coincide with each in turn. Then the edge is said to have described angles whose sum is four right angles. The same argument proves the angle-sum of spherical triangles to be two right angles, a result which we know to be wrong. It is to be regretted that in *two* American text-books, just published, this argument is reproduced, thus giving this famous heresy new life. For the exposition of the fallacy see G. B. Halsted's 4th Ed. of Bolyai's *Science Absolute of Space*, pp. 63–71; *Nature*, Vol. XXIX., 1884, p. 453.

[2] Engel and Stäckel, pp. 212, 213.

did not admit of proof. This scepticism appears in various writings.[1]

If it required courage on the part of Euclid to place his parallel-postulate, so decidedly unaxiomatic, among his other postulates and common notions, even greater courage was demanded to discard a postulate which for two thousand years had been the main corner stone of the geometric structure. Yet several thinkers of the eighteenth and nineteenth centuries displayed that independence of thought, so essential to great discoveries.

While Legendre still endeavoured to establish the parallel-postulate by rigorous proof, *Nicholaus Ivanovitch Lobatchewsky* (1793–1856) brought out a publication which assumed the contradictory of that axiom. Lobatchewsky's views on the foundation of geometry were first made public in a discourse before the physical and mathematical faculty of the University of Kasan (of which he was then rector), and first printed in the Kasan *Messenger* for 1829, and then in the *Gelehrte Schriften der Universität Kasan*, 1836–1838, under the title "New Elements of Geometry, with a complete theory of parallels." This work, in Russian, not only remained unknown to foreigners, but attracted no notice at home. In 1840 he published in Berlin a brief statement of his researches. The distinguishing feature of this "Imaginary Geometry" of Lobatchewsky is that through a point an indefinite number of lines can be drawn in a plane, none of which cut a given line in the plane, and that the sum of the angles in a triangle is variable, though always less than two right angles.[2]

[1] Engel and Stäckel, pp. 140, 141, 213–215.

[2] Lobatchewsky's *Theory of Parallels* has been translated into English by G. B. Halsted, Austin, 1891. It covers only forty pages. G. B. Halsted has also translated Professor A. Vasiliev's *Address* on the life and researches of Lobatchewsky, Austin, 1894.

A similar system of geometry was devised by the Bolyais in Hungary and called by them "absolute geometry." *Wolfgang Bolyai de Bolya* (1775–1856) studied at Jena, then at Göttingen, where he became intimate with young Gauss. Later, he became professor at the Reformed College of Maros-Vásárhely, where for forty-seven years he had for his pupils most of the present professors in Transylvania. Clad in old time planter's garb, he was truly original in his private life as well as in his mode of thinking. He was extremely modest. No monument, said he, should stand over his grave, only an apple-tree, in memory of the three apples: the two of Eve and Paris, which made hell out of earth, and that of Newton, which elevated the earth again into the circle of heavenly bodies.[1] His son, Johann Bolyai (1802–1860) was educated for the army, and distinguished himself as a profound mathematician, an impassioned violin-player, and an expert fencer. He once accepted the challenge of thirteen officers on condition that after each duel he might play a piece on his violin, and he vanquished them all.[2]

Wolfgang Bolyai's chief mathematical work, the *Tentamen*, appeared in two volumes, 1832–1833. The first volume is followed by an appendix written by his son Johann on *The Science Absolute of Space.* Its twenty-six pages make the name of Johann Bolyai immortal. Yet for thirty-six years this appendix, as also Lobatchewsky's researches, remained in almost entire oblivion. Finally Richard Baltzer, of Giessen, in 1867, called the attention of mathematicians to these won-

[1] F. Schmidt, "Aus dem Leben zweir ungarischer Mathematiker Johann und Wolfgang Bolyai von Bolya," Grunert's *Archiv der Mathematik und Physik*, 48 : 2, 1868.

[2] For additional biographical detail, see G. B. Halsted's translation of John Bolyai's *The Science Absolute of Space*, 4th Ed. 1896. Dr. Halsted is about to issue *The Life of Bolyai*, containing the Autobiography of Bolyai Farkas, and other interesting information.

derful studies. In 1894 a monumental stone was placed on the long-neglected grave of Johann Bolyai in Maros-Vásárhely. In the years 1893–1895 a Lobatchewsky fund was secured through contributions of scientific men in all countries, which goes partly toward founding an international prize for research in geometry, and partly towards erecting a bust of Lobatchewsky in the park in front of the university building in Kasan.

But the Russian and Hungarian mathematicians were not the only ones to whom the new geometry suggested itself. When Gauss saw the *Tentamen* of the elder Bolyai, his former room-mate at Göttingen, he was surprised to find worked out there what he himself had begun long before, only to leave it after him in his papers. His letters show that in 1799 he was still trying to *prove a priori* the reality of Euclid's system, but later he became convinced that this was impossible. Many writers, especially the Germans, assume that both Lobatchewsky and Bolyai were influenced and encouraged by Gauss, but no proof of this opinion has yet been presented.[1]

Recent historical investigation has shown that the theories of Lobatchewsky and Bolyai were, in part, anticipated by two writers of the eighteenth century, *Geronimo Saccheri* (1667–1733), a Jesuit father of Milan, and *Johann Heinrich Lambert* (1728–1777), a native of Mühlhausen, Alsace. Both made researches containing definitions of the three kinds of space, now called the non-Euclidean, Spherical, and the Euclidean geometries.[2]

[1] Engel and Stäckel, pp. 242, 243; G. B. Halsted, in *Science*, Sept. 6, 1895.

[2] The researches of Wallis, Saccheri, and Lambert, together with a full history of the parallel theory down to Gauss, are given by Engel and Stäckel. See also G. B. Halsted's "The non-Euclidean Geometry inevitable" in the *Monist*, July, 1894. Space does not permit us to speak of the later history of non-Euclidean geometry. We commend to our

We have touched upon the subject of non-Euclidean geometry, because of the great light which these studies have thrown upon the foundations of geometric theory. Thanks to these researches, no intelligent author of elementary text-books will now attempt to do what used to be attempted: *prove the parallel-postulate.* We know at last that such an attempt is futile. Moreover, many trains of reasoning are now easily recognized as fallacious by one who sees with the eyes of Lobatchewsky and Bolyai. Possessing, as we do, English translations of their epoch-making works, no progressive teacher of geometry in High School or College — certainly no author of a text-book on geometry — can afford to be ignorant of these results.

IV. *Text-books on Elementary Geometry.* — The history of the evolution of geometric text-books proceeds along different lines in the various European countries. About the time when Euclid was translated from the Arabic into Latin, such intense veneration came to be felt for the book that it was considered sacrilegious to modify anything therein. Still more pronounced was this feeling toward Aristotle. Thus we read of *Petrus Ramus* (1515–1572) in France being forbidden on pain of corporal punishment to teach or write against Aristotle. This royal mandate induced Ramus to devote himself to mathe-

readers Halsted's translations of Lobatchewsky and Bolyai. Georg Friedrich Bernhard Riemann's wonderful dissertation of 1854, entitled *Ueber die Hypothesen welche der Geometrie zu Grunde liegen* is translated by W. K. Clifford in *Nature*, Vol. 8, pp. 14–17, 36–37. Riemann developed the notion of n-ply extended magnitude. Helmholtz's lecture, entitled "Origin and Significance of Geometrical Axioms" is contained in a volume of *Popular Lectures on Scientific Subjects*, translated by E. Atkinson, New York, 1881. Consult also the works of W. K. Clifford. The bibliography of non-Euclidean geometry and hyperspace is given by G. B. Halsted in the *American Journal of Mathematics*, Vols. I. and II. Readers may be interested in *Flatland, a Romance of Many Dimensions*, published by Roberts Brothers, Boston, 1885.

matics; he brought out an edition of Euclid, and here again displayed his bold independence. He did not favour investigations on the foundations of geometry; he believed that it was not at all desirable to carry everything back to a few axioms; whatever is evident in itself, needs no proof. His opinion on mathematical questions carried great weight. His views respecting the basis of geometry controlled French text-books down to the nineteenth century. In no other civilized country has Euclid been so little respected as in France. A conspicuous example of French opinion on this matter is the text prepared by *Alexis Claude Clairaut* in 1741. He condemns the profusion of self-evident propositions, saying in his preface, "It is not surprising that Euclid should give himself the trouble to demonstrate that two circles which intersect have not the same centre; that a triangle situated within another has the sum of its sides smaller than that of the sides of the triangle which contains it. That geometer had to convince obstinate sophists, who gloried in denying the most evident truths . . .; but in our day things have changed face; all reasoning about what mere good sense decides in advance is now a pure waste of time and fitted only to obscure the truth and to disgust the reader." This book, precisely antipodal to Euclid, has contributed much toward moulding opinions on geometric teaching, but otherwise it did not enjoy a great success.[1] Similar views were entertained by *Étienne Bézout* (1730–1783), whose geometric works, like Clairaut's, are deficient in rigour. Somewhat more methodical was *Sylvestre François Lacroix* (1765–1843). But the French geometry which enjoyed the most pronounced success, both at home and in other countries, is that of *Adrien Marie Legendre*, first brought out in 1794. It is interesting to note

[1] See article "Géométrie" in the *Grand Dictionnaire Universal du XIX^e Siècle* par PIERRE LAROUSSE.

Loria's estimate of Legendre's *Éléments de géométrie.*[1] "They deserved it [success], since, as regards *form*, if they can compete with Euclid in clearness and precision of style, they are superior to him in the complex harmony which gives them the appearance of a beautiful edifice, divided into two symmetrical parts, assigned, the one to the geometry of the plane and the other to the geometry of space; and since in regard to *matter*, they surpass Euclid by being richer in material and better in certain particulars. But the great French analyst, in writing of geometry, could not forget his own favoured studies, so that in his hands geometry became a vassal of arithmetic, from which he borrowed a few ratiocinations and even some of his nomenclature, and took from its dominions the whole theory of proportions. If it is added that, while Euclid avoids the use of any figure, the construction of which is unknown to the reader, Legendre uses without scruple the so-called 'hypothetical constructions,' and that he gave the preference to that unfortunate definition of a straight line [used, moreover, even by Kant] as a minimum line; there is sufficient argument to support the fact that Legendre's edifice was not long in showing itself of a solidity incomparably inferior to its beauty."[2]

Thus we see that French writers, influenced by their views respecting methods of teaching, by their belief that what is apparent to the eye may be accepted by the young student without proof, and by their general desire to make geometry easier and more palatable, allowed themselves to depart a long way from the practice of Euclid. But Legendre is more strict than Clairaut; the reaction against Clairaut's views became more and more pronounced, and about the middle of the nine-

[1] Loria, *Della varia fortuna di Euclide*, Roma, 1893, p. 10.

[2] An American edition of Brewster's translation of Legendre's geometry was brought out in 1828 by Charles Davies of West Point. John Farrar of Harvard College issued a new translation in 1830.

teenth century, *Jean Marie Constant Duhamel* (1797–1872) and *J. Hoüel*[1] began to institute comparisons between Legendrian and Euclidean methods in favour of the latter, recommending the return to a modified Euclid. Duhamel proclaimed the method of limits to be the only rigorous one by which the use of the infinite can be introduced into geometry,[2] and contributed much toward imparting to that method a clear and unobjectionable form. Hoüel, professor at Bordeaux, while expressing his regret that Euclid's *Elements* had fallen into disuse in France, did not recommend the adoption of Euclid unmodified.

While Duhamel's and Hoüel's desired return to Euclidean methods did not take place, the discussion, nevertheless, led to improvements in geometric texts. The works which in France now enjoy the greatest favour are those of *E. Rouché* and *C. de Comberousse*, and of *Ch. Vacquant*.[3] The treatment of incommensurables is by the method of limits, against which the cry "lack of rigour" cannot be raised. In appendices considerable attention is given to projective geometry, but Loria remarks that the old and the new in geometry are here not united into a coherent whole, but merely mixed in disjointed fashion.[4]

Italy, the land where once Euclid was held in high veneration, where Saccheri wrote, in 1733, *Euclides ab omni naevo vindicatus* (Euclid vindicated from every flaw), finally discarded her Euclid and departed from the spirit of his *Elements*. In 1867 Professors Cremona of Milan and Battaglini of Naples

[1] Duhamel, *Des méthodes dans les sciences de raisonnement* (five volumes), 1866–1872, II., 326; Hoüel, *Essai sur les principes fondament aux de la géométrie élémentaire ou commentaire sur les XXXII. premières propositions des éléments d'Euclide* (Paris, 1867).

[2] Loria, *op. cit.*, p. 11. [3] Loria, *op. cit.*, p. 12.

[4] An elementary geometry of great merit, modelled after French works, particularly that of Rouché and Comberousse, was prepared in 1870 by William Chauvenet of Washington University in St. Louis.

were members of a special government commission to inquire into the state of geometrical teaching in Italy. They found it to be so unsatisfactory and the number of bad text-books so great and so much on the increase, that they recommended for classical schools the adoption of Euclid pure and simple, even though Cremona admitted that Euclid is faulty.[1] This recommendation became law. Later the use of Euclid's text was replaced by that of other works, prepared on analogous plans.[2] Thereupon appeared after the amended Euclidean model, as distinguished from the Legendrian, the meritorious work by A. Sannia and E. D'Ovidio. Later came the *Elementi di geometria* of Riccardo de Paolis and the *Fondamenti di geometria* of Guiseppe Veronese.

The high esteem in which Euclid was held in Germany during early times is demonstrated by many facts. About the close of the eighteenth century Abraham Gotthelf Kästner remarked that "the more recent works on geometry depart from Euclid, the more they lose in clearness and thoroughness." This points to a falling off from the Euclidean model. The most popular texts in use about the middle of the nineteenth century, and even those of the present time, are, from a scientific point of view, anything but satisfactory. Thus in H. B. Lübsen's *Elementar-Geometrie*, 14th Ed., Leipzig, 1870, written in the country which produced Gauss, and at a time when Lobatchewsky's and Bolyai's immortal works had been published 41 and 37 years respectively, we still find (p. 52) a proof of the parallel-postulate![3] If we bear in mind that geometry deals with continuous magnitudes, *in which com-*

[1] Hirst, address in *First Annual Report*, A. I. G. T., 1871.

[2] Loria, *op. cit.*, p. 15.

[3] Another illustration of the confusion which prevailed regarding the foundations of geometry, long after Lobatchewsky, is found in Thomas Perronnet Thompson's *Geometry without Axioms*, 3d Ed., 1830, in which he endeavoured to "get rid" of axioms and postulates!

mensurability is exceptional, it is somewhat startling that Lübsen should make no mention of incommensurables. Another work which has been used for several decennia is Karl Koppe's *Planimetrie*, 4th Ed., Essen, 1852. It speaks of parallels as having the same direction, and fails to consider incommensurability excepting once in a foot-note. Kambly's inferior work appeared in 1884 in the 74th edition.[1] The subject of geometric teaching has been much discussed in Germany. Some excellent text-books have been written, but they do not appear to be popular. Among the better works are those of Baltzer, Schlegel, H. Müller, Kruse, Worpitzky, Henrici and Treutlein.[2] Most of these seem to be dominated by Euclidean methods.[3] One of the questions debated in Germany and elsewhere is that pertaining to the rigidity of figures. While Euclid sometimes moves a figure as a whole, he never permits its parts to move relatively to one another. With him, figures are "rigid." In many modern works this practice is abandoned. For instance, we often allow an angle to be generated by rotation of a line,[4] whereby we arrive at the notion of a "straight angle" (not used by Euclid), which now competes with the right angle as a unit for angular measure. The abandonment of rigidity brings us in closer touch with the modern geometry, and is, we think, to be recommended, provided that we proceed with circumspection. Pure projective geometry needs neither motion nor rigidity.

[1] A. Ziwet in *Bulletin N. Y. Math. Soc.*, 1891, p. 6. Ziwet here reviews a book containing much information on the movement in Germany, viz. H. Schotten, *Inhalt und Methode des planimetrischen Unterrichts.* The reader will profit by consulting Hoffmann's *Zeitschrift für mathematischen und naturwissenschaftlichen Unterricht*, published by Teubner, Leipzig.

[2] Loria, p. 19.

[3] See Loria, p. 19, Schotten, *op. cit.*; H. Müller, *Besitzt die heutige Schulgeometrie noch die Vorzüge des Euklidischen Originals?* Metz, 1889.

[4] H. Müller, *op. cit.*, p. 2.

Preparatory courses in intuitive geometry, in connection with geometric drawing, have been widely recommended and adopted in Germany.

England has been the home of conservatism in geometric teaching. Wherever in England geometry has been taught, Euclid has been held in high esteem. It appears that the first to rescue the *Elements* from the Moors and to bring it out in Latin translation was an Englishman. The first English translation (Billingsley's) was printed in 1570. Before this, Robert Recorde made a translation, but it was probably never published.[1] Elsewhere we have spoken of mediæval geometrical teaching at Oxford. It was carried on with very limited success. About 1620 *Sir Henry Savile* (1549–1622), warden of Merton College, endeavoured to create an interest in mathematical studies by giving a course of lectures on Greek geometry. These were published in 1621. On concluding the course he used the following language: "By the grace of God, gentlemen hearers, I have performed my promise; I have redeemed my pledge. I have explained, according to my ability, the definitions, postulates, axioms, and the *first eight propositions* of the *Elements* of Euclid. Here, sinking under the weight of years, I lay down my art and my instruments."[2] It must be remembered that at this time scholastic learning and polemical divinity held sway. Savile ranks among those who laboured for the revival of true knowledge. He founded two professorships at Oxford, one of geometry and the other of astronomy, and endowed each with a salary of £150 per annum. In the preamble to the deed by which he annexed this salary he says that geometry was almost totally abandoned and unknown in England.

[1] W. W. R. Ball, *Maths. at Cambridge*, p. 18.

[2] Translated from the Latin by W. Whewell, in his *Hist. of the Induct. Sciences*, Vol. I., New York, 1858, p. 205.

Savile delivered thirteen lectures in his course, in which philological and historical questions received more attention than did geometry itself.[1] In one place he says, "In the beautiful structure of geometry there are two blemishes, two defects; I know no more." These "blemishes" are the theory of parallels and the theory of proportion. At that time Euclid's parallel-postulate was objected to as unaxiomatic and requiring demonstration. We now know by the light derived from non-Euclidean geometry that it is a pure assumption, incapable of proof; that a geometry exists which assumes its contradictory; that it separates the Euclidean from the pseudo-spherical geometry. No "blemish," therefore, exists on this point in Euclid's *Elements*. The second "blemish" referred to the sixth statement in Book V.[2] Pedagogically, this fifth book is still criticised, on account of its excessive abstruseness for young minds; but scientifically (aside from certain unimportant emendations which Robert Simson thought necessary) this book is now regarded as one of exceptional merit.

During the seventeenth and eighteenth centuries a considerable number of English editions of Euclid were brought out. These were supplanted after 1756 by Robert Simson's *Euclid*. In the early part of the nineteenth century Euclid was still without a rival in Great Britain, but the desirability of modification arose in some minds. As early as 1795, *John Playfair* (1748–1819) brought out at Edinburgh his *Elements of Geometry*, containing the first six books of Euclid and a supplement, embracing an approximate quadrature of the circle (*i.e.* the computation of π) and a book on solid geometry drawn from other sources. In the fifth book of Euclid, Playfair endeavours to remove Euclid's diffuseness of style by

[1] Cantor, II., 364. [2] Engel and Stäckel, p. 47.

introducing the language of algebra. Playfair also introduces a new parallel-postulate, simpler than Euclid's. Gradually the advantage of more pronounced deviations from Euclid's text was expressed. In 1849 De Morgan pointed out defects in Euclid.[1] About twenty years later Wilson[2] and Jones[3] expressed the desirability of abandoning the Euclidean method. Finally in 1869 one of the two greatest mathematicians in England raised his powerful voice against Euclid. J. J. Sylvester said, "I should rejoice to see . . . Euclid honourably shelved or buried 'deeper than e'er plummet sounded' out of the schoolboy's reach."[4] These attacks on Euclid as a schoolbook brought about no immediate change in geometrical teaching. The occasional use of Brewster's translation of Legendre or of Wilson's geometry no more indicated a departure from Euclid than did the occasional use, in the eighteenth century,

[1] Loria, *op. cit.*, p. 24, refers to De Morgan in the *Companion to the British Almanac*, which we have not seen.

[2] *Educational Times*, 1868.

[3] *On the Unsuitableness of Euclid as a Text-book of Geometry*, London.

[4] Sylvester's *Presidential Address to the Math. and Phys. Section of the Brit. Ass. at Exeter*, 1869, given in Sylvester's *Laws of Verse*, London, 1870, p. 120. This eloquent address is a powerful answer to Huxley's allegation that "Mathematics is that study which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation." We quote the following sentences from Sylvester: "I, of course, am not so absurd as to maintain that the habit of observation of external nature will be best or in any degree cultivated by the study of mathematics." "Most, if not all, of the great ideas of modern mathematics have had their origin in observation." "Lagrange, than whom no greater authority could be quoted, has expressed emphatically his belief in the importance to the mathematician of the faculty of observation; Gauss called mathematics a science of the eye . . . ; the ever to be lamented Riemann has written a thesis to show that the basis of our conception of space is purely empirical, and our knowledge of its laws the result of observation, that other kinds of space might be conceived to exist subject to laws different from those which govern the actual space in which we are immersed."

of Thomas Simpson's geometry. One happy event, however, grew out of these discussions, the organization, in 1870, of the *Association for the Improvement of Geometrical Teaching* (A. I. G. T.). T. A. Hirst, its first president, expressed himself as follows: "I may say further, that I know no successful teacher who will not admit that his success is almost in proportion to the liberty he gives himself to depart from the strict line of Euclid's *Elements* and to give the subject a life which, without that departure, it could not possess. I know no geometer who has read Euclid *critically*, no teacher who has paid attention to modes of exposition, who does not admit that Euclid's *Elements* are full of defects. They 'swarm with faults' in fact, as was said by the eminent professor of this college (De Morgan), who has helped to train, perhaps, some of the most vigorous thinkers of our time."[1]

After much discussion the Association published in 1875 a *Syllabus* of Plane Geometry, corresponding to Books I.–VI. of Euclid. It was the aim "to preserve the spirit and essentials of style of Euclid's *Elements;* and, while sacrificing nothing in rigour either of substance or form, to supply acknowledged deficiences and remedy many minor defects. The sequence of propositions, while it differs considerably from that of Euclid, does so chiefly by bringing the propositions closer to the fundamental axioms on which they are based; and thus it does not *conflict* with Euclid's sequence in the sense of proving any theorem by means of one which follows it in Euclid's order, though in many cases it simplifies Euclid's proofs by using a few theorems which are contained in the sequence of the *Syllabus*, but are not explicitly given by Euclid."[2] The *Syllabus* received the careful consideration of the best mathe-

[1] *First General Report*, A. I. G. T., 1871, p. 9.
[2] *Thirteenth General Report*, 1887, pp. 22–23.

matical minds in England; the British Association for the Advancement of Science appointed a committee to examine the *Syllabus* and to co-operate with the A. I. G. T. In their report, in 1877, which favoured the *Syllabus*, was expressed the conviction that "no text-book that has yet been produced is fit to succeed Euclid in the position of authority." One of the great drawbacks to reform was the fact that the great universities of Oxford and Cambridge in their examinations for admission insisted upon a rigid adherence to the proofs and the sequence of propositions as given by Euclid; thus no freedom was given to teachers to deviate from Euclid in any way. To be noted, moreover, was the total absence of "originals" or "riders," or any question designed to determine the real knowledge of the pupil. But in the last few years the work of the Association has been recognized by the universities, and some freedom has been granted. It was noted above that J. J. Sylvester was an enthusiastic supporter of reform. The difference in attitude on this question between the two foremost British mathematicians, J. J. Sylvester, the algebraist, and Arthur Cayley, the algebraist and geometer, was grotesque. Sylvester wished to bury Euclid "deeper than e'er plummet sounded" out of the schoolboy's reach; Cayley, an ardent admirer of Euclid, desired the retention of Simson's *Euclid*. When reminded that this treatise was a mixture of Euclid and Simson, Cayley suggested striking out Simson's additions and keeping strictly to the original treatise.[1]

The most difficult task in preparing the *Syllabus* was the treatment of proportion, Euclid's Book V. The great admiration for this Book V. has been inconsistent with the practice in the school-room. Says Nixon, "the present custom of *omitting* Book V., though quietly assuming such of its results

[1] *Fifteenth General Report*, A. I. G. T., 1889, p. 21. See also *Fourteenth General Report*, p. 28.

as are needed in Book VI., is singularly illogical."[1] Hirst offers similar testimony: "The fifth book of Euclid . . . has invariably been 'skipped,' by all but the cleverest schoolboys."[2] Here we have the extraordinary spectacle of proportion relating to magnitudes "skipped," by consent of teachers who were shocked at Legendre, because he refers the student to arithmetic for his theory of proportion! Is not this practice of English teachers a tacit acknowledgment of the truth of Raumer's[3] assertion that as an elementary textbook the *Elements* should be rejected? Euclid himself probably never intended them for the use of beginners. But the English are not prepared to follow the example of other nations and cast Euclid aside. The British mind advances by evolution, not by revolution. The British idea is to *revise*, *simplify*, and *enrich* the text of Euclid.

The first important attempt to revise and simplify the fifth book was made by Augustus De Morgan. He published, in 1836, "The Connexion of Number and Magnitude: an attempt to explain the fifth book of Euclid." In 1837 it appeared as an appendix to his "Elements of Trigonometry." It is on this revision that the substitute for Book V., given in the *Syllabus*, is modelled. On the treatment of proportion there was great diversity of opinion among the members of the subcommittee appointed by the Association to prepare this part of the *Syllabus*, and no unanimous agreement was reached.[4] There were, in the first place, different schemes for re-arranging and simplifying Euclid's Book V. According to Euclid four magnitudes, a, b, c, d, are in proportion, when for any integers

[1] R. C. J. Nixon, *Euclid Revised*, Oxford, 1886, p. 9.

[2] *Fourth General Report*, A. I. G. T., 1874, p. 18.

[3] *Geschichte der Pädagogik*, Vol. III.; see also K. A. Schmid, *Encyklopädie des gesammten Erziehungs- und Unterrichtswesens*, art. "Geometrie."

[4] Nixon, *op. cit.*, p. 9.

m and n, we have simultaneously $ma \gtreqless nb$, and $mc \gtreqless nd$. A definition of proportion adopted by the Italians Sannia and D' Ovidio was considered, according to which parts are substituted for multiples, viz. a, b, c, d, are in proportion, if, for any integer m, a contains $\frac{b}{m}$ neither a greater nor a less number of times than c contains $\frac{d}{m}$. Another method discussed was that of approaching incommensurables by the theory of limits, as adopted in the French work of E. Rouché and C. de Comberousse, and by many American writers. This method, as also Sannia and D' Ovidio's definition of proportion, rests on the law of continuity which Clifford defines *roughly* as meaning "that all quantities can be divided into any number of equal parts."[1] If in a definition it is desirable to assume as little as possible, Euclid's definition of proportion is preferable, inasmuch as it does not postulate this law. To minds capable of grasping Euclid's theory of proportion, that theory excels in beauty the treatment of incommensurables with aid of the theory of limits. Says Hankel, "We cannot suppress the remark that the now prevalent treatment of irrational magnitudes in geometry is not well adapted to the subject, in as much as it separates in the most unnatural manner things which belong together, and forces the continuous — to which, from its very nature, the geometric structure belongs — into the shackles of the discrete, which nevertheless it every moment pulls asunder."[2]

About the time of Hankel a new logical difficulty came to light. It became evident that the theory of limits as it existed in the first half of the nineteenth century must remain unsatisfactory unless a theory of irrational numbers could

[1] *The Common Sense of the Exact Sciences*, New York, 1891, p. 107.
[2] HANKEL, *op. cit.*, p. 65.

be presupposed. As it was, the lines of procedure involved reasoning in a circle. The theory of limits was invoked to establish a theory of irrational numbers, whereupon the theory of irrational numbers seemed to be used for a more complete exposition of the theory of limits. To avoid this vicious circle, altogether new theories of irrational numbers were developed by Karl Weierstrass, G. Cantor, and R. Dedekind, which do not pre-require the theory of limits.[1] These developments have greatly strengthened the logical foundations of mathematics, but have not yet been put in a form suitable for use in elementary instruction. From the standpoint of the elementary teacher, a logical exposition of irrational numbers and of limits, appears impossible. Neither Euclid's treatment of proportion, nor the treatment of irrational number and limits as given by modern logicians, is proper food for school children. The quest for perfect logical rigor is long and disheartening. Indeed it is doubtful whether mathematicians have yet reached absolute rigor. The creations of G. Cantor and Dedekind have only recently given rise to contradictory results, to antinomies, that are thoroughly disquieting. To be sure, the introduction of certain restrictions in the definitions and the adoption of certain "axioms" have removed the antinomies advanced thus far. But we know not the day nor the hour when new antinomies will spring to light. In view of the subtlety of the modern theories of numbers and of doubt as to their real logical perfection, there remains no alternative for the elementary teacher but to reconcile himself with a logically imperfect treatment of the theory of limits and of irrational

[1] Consult Georg Cantor, *Contributions to the Founding of the Theory of Transfinite Numbers*, translated, and provided with an Introduction and Notes, by Philip E. B. Jourdain, Chicago and London, 1915. Jourdain's Introduction traces the history of the subject. Consult also George Bruce Halsted, *On the Foundation of Technic of Arithmetic*, Chicago, 1912.

numbers.[1] In fact it is generally felt that these subjects do not really belong to elementary mathematics, that a rudimentary and purely intuitive treatment of limits is all that should be attempted in high school and in the first stages of college mathematics.

The same conclusion on matters of logic is reached from a different angle. Italian mathematicians under the leadership of Giuseppe Peano of the University of Turin set themselves the problem to adopt definitions, axioms, and undefined terms so that geometry might be built up without appealing to intuition at any point. As is well known, Euclid quietly resorts to intuition in many places, for the knowledge of certain facts. In the very first proposition of his *Elements* he states without proof that the two circles which he draws intersect each other. As contributions to the logical exposition of geometry, the work of the Italian school was praiseworthy. A culmination of this movement is found in a treatise prepared by the great German representative of logical abstract thought in mathematics, David Hilbert of the University of Göttingen. At the unveiling of the Gauss-Weber monument in Göttingen, in June, 1899, Hilbert contributed a memorial address on the foundations of Geometry, entitled *Grundlagen der Geometrie*, which was widely recognized as a masterpiece of logical exposition. The first edition was translated into English in 1902 by E. J. Townsend of the University of Illinois. Several German editions, somewhat enlarged, have appeared since. Hilbert sets up five groups of axioms, which include altogether twenty axioms. To show how Hilbert's axioms can serve as a foundation of geometry, built up without appeal to intuition, George Bruce Halsted published in 1904 his *Rational Geometry*. This book affords perhaps the easiest approach to a knowledge of Hilbert's

[1] For an illuminating article on limits, consult Bertrand Russell's article, "Mr. Haldane on Infinity," in *Mind*, Vol. 33, London, 1908, p. 239.

system. At the same time it demonstrates the utter impossibility of introducing high school students into geometry by such a route. To meet certain logical criticisms of his text, Halsted prepared a thoroughly revised second edition in 1907, which has been translated into French.

RECENT MOVEMENTS IN TEACHING

Toward the close of the nineteenth century the influence of the Association for the Advancement of Geometric Teaching (A. I. G. T.), whose name was changed in 1897 to "The Mathematical Association" and whose work had been broadened in scope, so as to include elementary mathematical instruction in general, showed itself in many ways. To note the progress in geometry we need only examine such editions and revisions of Euclid as those of Casey, Nixon, Mackay, Langley and Phillips, Taylor, and the *Elements of Plane Geometry* issued by the Association.[1]

At the beginning of the twentieth century there were two great movements in the teaching of elementary mathematics. One has been called the "Perry Movement," the other is a co-operative movement of teachers of different nations under the name of the "International Commission on the Teaching of Mathematics." Of these movements the first made at once a deep and lasting impress, especially in England and America, and then spent itself; the second was planned on a large scale and was still in the preliminary stages when interrupted by the great war which broke out in 1914.

[1] For the history of geometrical teaching in the United States before 1888, see Cajori's *The Teaching and History of Mathematics in the U. S.*, 1890.

The Perry Movement

The reform which this movement introduced is the culmination of thirty years of discussion in England, carried on by the members of the Mathematical Association. This culmination centers around the personality of John Perry, then Professor of mechanics and mathematics of the Royal College of Science, London. Perry was member of a large and strong committee of the British Association for the Advancement of Science, appointed in 1901 to report on the "Teaching of Mathematics." This was not the first time that the British Association had interested itself in this matter. A similar committee had reported on "the possibility of improving the methods of instruction in elementary geometry" in 1873 and again in 1876.[1] These early reports considered the advisability of adopting an authorized syllabus of geometry; the second report considered the merits of the syllabus prepared by the A. A. G. T. and recommended it; nevertheless that syllabus had not been generally adopted at the beginning of the new century. Teaching continued to be guided mainly by the requirements of examinations.

It was at the Glasgow meeting of the British Association in 1901, when Section A (Mathematics and Physics) and Section L (Education) held a joint meeting, that John Perry delivered a great address, which was followed by a brilliant discussion, leading to epoch-making results. An account of these deliberations was published in 1901 under the title, *Discussion on the Teaching of Mathematics*, edited by John Perry. This debate is far more spicy, interesting and valuable than the formal report of the committee, made a year later. Perry called for a complete divorce from Euclid, for greater emphasis

[1] See *Reports* of the British Association for the year 1873, page 459, and for 1876, page 8.

upon experimental geometry, for practical mensuration, for the use of squared paper, for more solid geometry, for greater emphasis upon the utilitarian parts of geometry. As a result, Euclid was dethroned in England from its position of complete dominance as a text-book in elementary schools. In closing the discussion, Perry said:

"I take it that we are all agreed upon the following points:

"1. Experimental methods in mensuration and geometry ought to precede demonstrative geometry, but even in the earliest stages some deductive reasoning ought to be introduced.

"2. The experimental methods adopted may greatly be left to the judgment of the teacher: they may include all those mentioned in the Elementary Syllabus which I presented. . . .

"3. Decimals ought to be used in arithmetic from the beginning.

"4. The numerical evaluation of complex mathematical expressions may be taken up almost as part of arithmetic or at the beginning of the study of algebra, as it is useful in familiarizing boys with the meaning of mathematical symbols.

"5. Logarithms may be used in numerical calculation as soon as a boy knows $a^n \times a^m = a^{m+n}$, and long before he is able to calculate logarithms. But a boy ought to have a clear notion of what is meant by the logarithm of a number.

"6. In mathematical teaching a thoughtful teacher may be encouraged to distinguish what is essential for education in the sequence which he employs, from what is merely according to arbitrary fashion, and to endeavor to find out what sequence is best, educationally, for the particular kind of boy whom he has to teach.

"7. Examination cannot be done away with in England. Great thoughtfulness and experience are necessary qualifications for an external examiner. It ought to be understood that an examination of a good teacher's pupils by any other

examiner than the teacher himself is an imperfect examination. . . .

"About these that follow I am in more doubt. . . .

"8. A thoughtful teacher ought to know that by the use of squared paper and easy algebra, by illustrations from dynamics and laboratory experiments, it is possible to give to young boys the notions underlying the methods of the Infinitesimal calculus.

"9. A thoughtful teacher may freely use the ideas and symbolism of the calculus in teaching elementary mechanics to students.

"10. A thoughtful teacher may allow boys to begin the formal study of the calculus before he has taken up advanced algebra or advanced trigonometry, or the formal study of analytical or geometrical conics, and ought to be encouraged to use in this study, not merely geometrical illustrations, but illustrations from mechanics and physics, and illustrations from any other quantitative study in which a boy may be engaged."

In the United States the Perry movement did not bring about as many changes as in England. There are two principal reasons for this. One reason is that Euclid had been discarded long ago as a text for beginners, hence the agitation in Great Britain on that point had no bearing on American conditions. A second reason is that some of the other recommendations of Perry did not directly fit the American situation. For example, Perry's warfare against the system of examinations then in vogue in British universities and also in the English civil service did not affect America, where examinations were conducted differently and received less stress. The clearest American declaration with respect to the Perry movement, in so far as it affected the United States, came from E. H. Moore of the University of Chicago in his presi-

dential address, "On the Foundations of Mathematics," delivered on December 29, 1902, before the American Mathematical Society.[1] Moore had in mind mainly college mathematics, when he said: "In agreement with Perry, it would seem possible that the student might be brought into vital relation with the fundamental elements of trigonometry, analytic geometry, and the calculus, on condition that the whole treatment in its origin is and in its development remains closely associated with thoroughly concrete phenomena." With reference to secondary schools, Moore puts the question: "Would it not be possible to organize the algebra, geometry, and physics of the secondary school into a thoroughly coherent four years' course, comparable in strength and closeness of structure with the four years' course in Latin?" As to method of instruction: "This program of reform calls for the development of a thoroughgoing laboratory system of instruction in mathematics and physics, a principal purpose being as far as possible to develop on the part of every student the true spirit of research, and an appreciation, practical as well as theoretic, of the fundamental methods of science."

Experience seems to show that the success of the laboratory method of mathematical instruction rests upon several contingencies, such as, (1) rearrangement of the schedule so as to allow more time in school for the courses in mathematics, (2) the equipment of suitable laboratories, (3) the enlistment of teachers who are competent to give laboratory courses, and so on. Even if all these conditions prevail, there is no *a priori* reason why the laboratory method should excel other methods. The superiority of any method over its competitors must be determined by actual trial. Such superiority has not yet been established experimentally for the laboratory method.

[1] This address was printed in *Bulletin of the Am. Math. Society*, Vol. 9, 1903, pp. 402–424, also in *Science*, N. S., Vol. 17, 1903, pp. 401–416.

We pass to the broader question, the unification of the different branches of high school mathematics, carried on in a manner that will meet more effectively the practical demands of the age. This question has been under discussion and experimentation ever since the opening of the new century. The impression seems to prevail that, in past decades and past centuries, mathematics never were unified, that teachers placed mathematical topics into separate and air-tight compartments. Nothing can be further from the truth. Previous to the nineteenth century there existed a fair degree of fusion of different branches of mathematics. We see this, for instance, in the Ahmes papyrus, in Euclid's *Elements*, with its three books on arithmetic, in the works of Vieta, Stevin, Oughtred, Newton, Leibniz, and Euler.

The geometric phraseology "linear" equation, "quadratic" equation, "cubic" equation, "a square" for a^2, "a cube" for a^3, shows that, in its formative period, algebra was not dissociated from geometry. Moreover, the solution of cubic and quartic equations by the method of geometric construction appears earlier in the history of mathematics than do the purely algebraic solutions. Isolation in mathematics as a science is a product of the nineteenth century. Steiner and Von Staudt isolated modern geometry from analysis. On the other hand, Weierstrass, Kronecker, Klein, and others worked in the direction of arithmetization which makes number the sole foundation of mathematics. This isolation is due to the efforts toward increased rigor of presentation, in course of which the foundations of mathematics were examined anew. Not only in the growth of mathematics as a science, but also in the teaching of mathematics, isolation of topics set in. Frequently, books which in their first editions exhibited a certain degree of fusion of topics, passed to greater isolation in later editions. Thus, Legendre's *Geometry*, first published in 1794, stood at

first for a moderate unification of geometry with arithmetic and trigonometry. As late as 1845 this text still contained trigonometry, but later the trigonometry and the practical applications of geometry were gradually filtered out. Nicholas Pike's *New and Complete System of Arithmetic*, printed at Newburyport, Mass., in 1788, and used for a time even in our American colleges, contained in addition to the ordinary arithmetical topics, a treatment of logarithms, trigonometry, algebra, and conic sections. The fourth edition of Pike's book omitted the topics not strictly arithmetical, and constituted Pike's *Abridged Arithmetic*. Thus there arose in teaching an isolation of topics, a moderate degree of which was beneficial. Teachers found it easier to reach good results in the classroom by taking one topic at a time. But isolation came to be carried to an extreme. The exercises of the schoolroom related to isolated topics. The problems of practical life called for ability to combine arithmetic, algebra, and geometry. The pupils lacked experience and skill in combination. Hence the need for reunification. No agreement has yet been reached as to the kind and degree of unification that is most suitable under modern conditions of industry and life. There is danger of excessive unification. Too much fusion may produce confusion. In intense fusion there is danger of introducing simultaneously a large number of new ideas, new definitions. The pupil does not concentrate upon a few ideas, but scatters his efforts over a large number of them and obtains only a superficial view of any one.

If it be asked what degree of unification can be safely adopted as having stood the test of experience, the answer would seem to be as follows: At any stage of mathematical study, let the pupil use all the mathematics that he has had up to that time. For instance, when he studies algebra, let him use freely the arithmetic that preceded, as well as the intuitive geometry

previously learned; when pursuing solid geometry, let him use the arithmetic, algebra, and plane geometry previously studied. To this extent, unification is sanctioned by experience. How much farther we can go advantageously under modern conditions remains to be determined by experience. Doubtless some allowance must be made for individual predilections of teachers and the practical needs of different pupils.

The International Commission

At the Fourth International Congress of mathematicians that was held at Rome in 1908, David Eugene Smith of Teachers College, Columbia University, made a motion calling for the appointment of an International Commission on the Teaching of Mathematics. The motion was carried and a Central Committee was appointed, consisting of Felix Klein of the University of Göttingen, Sir George Greenhill of London, and Henri Fehr of the University of Geneva in Switzerland. In 1912 David Eugene Smith was added to the committee. The Commission adopted as its official organ *L'Enseignement mathématique, Revue internationale paraissant tous les deux mois.* This bi-monthly journal had been started at Paris in 1899; it appears under the editorship of C. A. Laisant and H. Fehr. Thus far the Commission has chiefly occupied itself in effecting organizations in the different countries, and in collecting and publishing, for comparative examination, the fullest possible information as to the present status of the teaching of mathematics in each of the countries represented. The first report of the International Commission was made at the Fifth International Congress of Mathematicians,[1] held at Cambridge, England, in 1912. At that time there were 27 countries represented, and 150 printed reports were made, leaving about 50

[1] For an account of the work of this congress, see J. W. A. Young in *School Science and Mathematics*, Vol. 12, 1912, pp. 702–715.

still to be published. These reports contain detailed information on the condition of mathematical teaching at different schools and in different countries. They are important contributions to the history of education. Most exhaustive are the numerous reports prepared by the Germans. The British reports are models of concise presentation. The documents prepared in the United States by special committees acting under the direction of the American Commissioners (D. E. Smith of Teachers College, W. F. Osgood of Harvard University, J. W. A. Young of the University of Chicago) have been published by the United States Bureau of Education. The breaking out of the Great War, in 1914, interrupted the work of the International Commission.

While the publications[1] hitherto issued are mainly historical, and descriptive of present conditions, it is evident that the representatives of the international movement stand for the more thorough embodiment, in mathematical teaching, of certain ideas, at least a part of which were emphasized in the Perry movement. Predominant are the ideas of graphic representation, functionality, conservative unification of mathematics, alignment with the practical demands of the age. In Germany these ideas were strongly advocated by F. Klein at a school conference held in 1900. In France these same ideas found expression in the text-books published by E. Borel in 1903, which were translated into German by P. Stäckel in

[1] The French reports are entitled, *Rapports de la Commission Internationale de l'Enseignement Mathématique*, Hachette & Company, Paris. The English reports are brought out under the heading *The Teaching of Mathematics in the United Kingdom, being a series of papers prepared for the International Commission on the teaching of mathematics*, Wyman & Sons, Fetter Lane, London, E. C. The German reports are *Abhandlungen über den mathematischen Unterricht in Deutschland, Veranlasst durch die Internationale mathematische Unterrichtskommission.* Herausgegeben von F. Klein, Vols. I–V, B. G. Teubner, Leipzic u. Berlin.

1908, to serve as illustrations of what can be done in the way of actual realization of the new plans.

The subject of graphic representation of mathematical expressions involving variables goes back to the time of Newton and Descartes; the graphic representation of curves defined by certain properties goes back to the time of the Greeks.

In the nineteenth century graphic methods came to be used widely in mechanical science,[1] not only abroad but also in the United States. In 1898 R. H. Thurston of Cornell University read before the American Society of Mechanical Engineers a paper on "Graphic Diagrams and Glyptic Models." In the teaching of elementary mathematics graphic methods were introduced slowly. Lone voices advocating the use of graphs in teaching were heard in the nineteenth century.

In 1842 Strehlke of Danzig expressed himself as follows:[2] "The graphic representation of functions appears to me an effective aid in instruction, for the purpose of conveying a clear conception of their properties. With this end in view I have drawn a considerable number of such graphs — the curves of sines, cosines, tangents, of natural and Briggian logarithms. . . . Even simple functions like $y = x$, $y = x^2$, $y = x^3$, $y = x^4$, if they are drawn for the same variable and upon the same sheet, present properties which the pupil would not see the moment the function is put before him." In the solution of equations, graphic methods received recognition in L. Saint-Loup's *Traité de la résolution des équations numériques*, Paris, 1861, a text written for students in the Polytechnic School and the Normal School in Paris, "à l'usage des candidats aux écoles polytechnique et normale." The students were

[1] See the "Second Report on the Development of Graphic Methods in Mechanical Science," by H. S. Hele Shaw, in *Report of the British Association for the Advancement of Science* for the year 1892, pp. 373–531.

[2] Grunert's *Archiv*, Vol. 2, 1842, p. 111.

more advanced than are American students of high school grade. In the *Tutorial Trigonometry*, by William Briggs and G. H. Bryan, London, 1897, attention is given to the tracing of the curves of the trigonometric functions.

In 1896 appeared Arthur H. Barker's *Graphical Calculus*, London, which represented a series of lectures given to engineering students. It marks a complete contrast to the ordinary mathematical book of the school of Todhunter. It is a work intended for students ready to enter upon the study of the calculus.

In the United States the earliest use of graphic methods in elementary teaching consisted in the representation, by curves, of empirical data, such as temperatures, cloudiness of the sky. From 1885 to 1889 such methods were taught in the school of the Chautauqua Town and Country Club, to about two thousand pupils.[1] But these graphic methods met with no general adoption at that time. In 1898 Francis E. Nipher of Washington University in St. Louis brought out an *Introduction to Graphic Algebra*. The first regular text-book using graphs was the *Algebra for Schools* (1899) by George W. Evans of Boston. In 1902 there appeared as a supplement to Webster Wells' *Essentials of Algebra* and *New Higher Algebra*, a monograph of twenty-one pages on *Graphs*, prepared by Robert J. Aley, then of the University of Indiana. Henry B. Newson of the University of Kansas followed in 1905 with a *Graphic Algebra for Secondary Schools*. In Vol. 5 (1905) of *School Science and Mathematics* there is an article on graphs by F. C. Touton of Kansas City. Since 1905 our algebras for secondary schools introduce the graph as a regular subject for instruction. As early as 1903 "the use of the graphic methods

[1] See CHARLES BARNARD, "Graphic Methods in Teaching," *Educational Monographs published by the N. Y. College for the Training of Teachers*, Vol. II., No. 6, 1889, pp. 209–223.

and illustrations, particularly with the solution of equations, is also expected" as an entrance requirement in mathematics to eastern colleges.

American Associations

There have been organized in the United States a number of associations for the advancement of the teaching of mathematics and science in secondary schools. Thus we have the Association of Teachers of Mathematics in New England, the Association of Ohio Teachers of Mathematics and Science, the Association of Teachers of Mathematics in the Middle States and Maryland, Central Association of Science and Mathematics Teachers, Colorado Mathematics Association, Kansas Association of Mathematics, Missouri Society of Teachers of Mathematics and Science, Iowa Association of Mathematics Teachers, North Dakota Association of Science and Mathematics Teachers, Southern California Science and Mathematics Association, Oregon State Science and Mathematical Teachers Association.

The Mathematical Association of America, organized in December, 1914, devotes itself mainly to the teaching of Mathematics in College. The American Mathematical Society (organized in 1888 as the New York Mathematical Society, and reorganized under its present name in 1894) devotes itself entirely to the furtherance of original research.

The National Educational Association (organized in Philadelphia in 1857 under the name of "National Teachers' Association" and re-named in 1870) has always interested itself in the teaching of elementary mathematics. In 1893 a "Committee of Ten" reported on secondary school studies, including the recommendations of various sub-committees or "conferences." A "conference of ten" on mathematics met at Harvard in December, 1892. Simon Newcomb acted as chairman.

A report was prepared, which is given in outline in the *Report of the Commissioner of Education*, 1892–1893, Vol. 2, p. 1426. Full reports of the "conferences" were published separately in 1894. It was recommended by the mathematics conference that the course in arithmetic be abridged by omitting subjects which perplex and exhaust the pupil without affording mental discipline, and enriched by exercises in concrete problems, that geometry be first taught in concrete form, including measurements, and that greater care be taken to avoid slovenliness of expression in demonstrations. Stress was laid upon the interlacing of arithmetic, algebra, and geometry. The eagerness with which the various reports were received caused a "Committee of Fifteen" to be appointed by the National Education Association to institute similar inquiries for schools below the high school. Recommendations were made on the teaching of arithmetic.[1] It was argued that the simplest fractions, $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$, could be taught very early, that compound interest, foreign exchange, and certain other topics be dropped, that mental arithmetic be emphasized, that there be a rearrangement of subjects, and that the rudiments of algebra be introduced in the seventh year.

In 1899 the Committee on College Entrance Requirements, appointed by the N. E. A., made recommendations in mathematics which were in practical agreement with those of the Committee of Ten. In 1904 the Association of Mathematics Teachers in New England appointed a committee, with George W. Evans of Boston as chairman, to consider the advisability of preparing a geometry syllabus, or adopting some syllabus already drawn up. Such a syllabus was drawn up; though carefully prepared, it did not become generally known. A report by a syllabus committee of the Central Association of Science and Mathematics Teachers, made in 1906, attracted

[1] *Report of the U. S. Commissioner of Education for* 1893–1894, p. 532.

wider attention, but was imperfect in some respects. Finally, in 1908 and 1909 a committee was organized under the joint auspices of the National Education Association and the American Federation of Teachers of the Mathematical and National Sciences. H. E. Slaught of the University of Chicago acted as chairman. The final report of this committee was printed in 1912, and received much attention both in the United States and abroad. Theorems are classified into four groups, according to their importance, the groups being distinguished one from the other by the kind of type used. The report gives considerable attention also to logical considerations, and contains an introduction on the history of the teaching of geometry during the last two centuries.

In recent time there has been considerable unrest in the United States on matters relating to mathematics in secondary schools. Proceeding on the theory that the real aim is not, as formerly held, to "train the mind," but to teach only "things that are practical," so that the mathematical instruction can be "hitched up to life," some educators exert pressure to omit certain subjects formerly thought essential, such as the treatment of factoring and fractions in algebra, and to restrict problems in algebra and exercises in geometry to those that are practical. In some ways the modern tendency resembles the movement of the fifteenth and sixteenth centuries, led by the "Commercial School of Arithmeticians." As in the sixteenth century, so now, there is meant by "practical," not that which is *ultimately* practical, but that which is *immediately* practical. By this test a course resembling ancient Egyptian geometry makes a stronger appeal than one resembling the more scientific Greek geometry. There are other modern educators who go farther and, denying the disciplinary value of mathematics, insist that high school mathematics, or at least a part of it, should be removed from the required list of high

school studies. It may be of interest, in this place, to call attention to earlier attacks on the value of mathematics as an exercise in the training of the mind.

Attacks upon the Study of Mathematics as a Training of the Mind.[1]

Probably the most famous attack that has ever been made upon the educational value of mathematics was published by the philosopher, Sir William Hamilton, in the *Edinburgh Review* of 1836. He claims that mathematics is "not an improving study"; "If we consult reason, experience and the common testimony of ancient and modern times, none of our intellectual studies tend to cultivate a smaller number of the faculties, in a more partial manner, than mathematics." He proceeds to adduce testimony to the effect that "the cultivation afforded by the mathematics is, in the highest degree, one-sided and contracted," that mathematics "freeze and parch the mind," that this science is "absolutely pernicious as a mean of internal culture." Hamilton's argument proceeds along two principal lines; the first is to prove that those who confine their studies to mathematics alone are addicted to blind credulity or irrational scepticism and, in general, lack good judgment in affairs of life. We admit at once, that this last contention is probably valid; the *exclusive* study of any branch of knowledge is to be discouraged as undesirable for a liberal education. Hamilton's second mode of attack consists in trying to show that many mathematicians found mathematics unsatisfactory as an exercise of the mind, and renounced it. The only mathematicians of sufficient prominence to be known

[1] See F. Cajori, "A review of three famous attacks upon the study of mathematics as a training of the mind," in the *Popular Science Monthly*, 1912, pp. 360–372. See also Dr. C. J. Keyser, *Mathematics*, Columbia Univ. Press, 1907, pp. 20–44.

to the modern reader are D'Alembert, Pascal, Descartes, and Dugald Stewart. Many other men, from all ages and nations, are quoted, and constitute indeed a "cloud of witnesses." Says Alfred Pringsheim of Munich: "I am ashamed to confess that before reading Hamilton's article I did not know a single one of these great authorities even by name; an extenuating circumstance is the fact that some of these names I could not find even in the scientific directories." The most extensive examination of Hamilton's quotations and most thorough disproof of his contentions are found in an article by A. T. Bledsoe in the *Southern Review* for July, 1877. Bledsoe was at one time professor of mathematics at the University of Virginia. He proves what some other writers before him hinted at, or proved only in part, namely, that Hamilton was extremely careless in the selection of his quotations. By making partial extracts, badly chosen, he made scientists say exactly the opposite of their real sentiments.[1] Most strikingly is this true of his treatment of Descartes.

The second great attack upon mathematical study was made by Schopenhauer, the pessimistic sage of Frankfort-on-the-Main. In his *The World as Will and Idea*, second edition, 1844, he makes war on Euclid and his geometrical demonstrations. "We are compelled by the principle of contradiction to admit that what Euclid demonstrates is true, but we do not comprehend *why* it is so. We have, therefore, almost the same uncomfortable feeling that we experience after a juggling trick, and, in fact, most of Euclid's demonstrations are remarkably like such feats. The truth almost always enters by the back door, for it manifests itself *par accidens* through some contingent circumstance. Often a *reductio ad absurdum* shuts all the doors one after another, until only one is left, which

[1] On this point consult also JOHN STUART MILL'S *An Examination of Sir William Hamilton's Philosophy*, New York, 1884, Vol. II., pp. 305–336.

we are therefore compelled to enter." This is part of Schopenhauer's famous characterization of mathematical reasoning as "mousetrap proofs" (Mausefallenbeweise). His objections are directed almost entirely against Euclid. He had no acquaintance with modern mathematics. His criticism of Euclid as a text for children is largely valid. Euclid did not write for children. It is a historical puzzle, difficult to explain, how Euclid ever came to be regarded as a text suitable for the first introduction into geometry. As to the *reductio ad absurdum* method of proof. De Morgan said: "The most serious embarrassment in the purely reasoning part is the *reductio ad absurdum*, or indirect demonstration. This form of argument is generally the last to be clearly understood, though it occurs almost on the threshold of the *elements*." Schopenhauer's attack bears only indirectly upon the question relating to the mind-training value of mathematics. His views must have attracted considerable attention in Germany, for as late as 1894 A. Pringsheim thought it necessary to refute his argument, and only nine years ago Felix Klein referred to him at some length in a mathematical lecture at the University of Göttingen.

A third attack upon mathematics was made in 1869 by the naturalist, Thomas H. Huxley. In the *Fortnightly Review*, 1869, p. 433, he said: Mathematics is that study "which knows nothing of observation, nothing of experiment, nothing of induction, nothing of causation." In an after-dinner speech, reported in *Macmillan's Magazine*, 1869, pp. 177–184, he argued against language and mathematical training, and in favor of scientific training. "Mathematical training," he said, "is almost purely deductive. The mathematician starts with a few propositions, the proof of which is so obvious that they are called self-evident, and the rest of his work consists of subtle deductions from them." Of great interest is the reply

made to Huxley by the mathematician, J. J. Sylvester.[1] To Americans Sylvester's name is memorable, because in 1876 Sylvester came over from England and for eight years taught modern higher algebra to American students at the Johns Hopkins University and gave a powerful stimulus to the study of higher mathematics in this country. Sylvester was an enthusiast. His reply to Huxley was his presidential address before Section A of the British Association in 1869. By his own experiences as a mathematician he tried to show that Huxley was wrong. Sylvester insists "that mathematical analysis is constantly invoking the aid of new principles, new ideas and new methods . . . springing direct from the inherent powers and activity of the human mind . . . that it is unceasingly calling forth the faculties of observation and comparison, that one of its principal weapons is induction, that it has frequent recourse to experimental trial and verification, and that it affords a boundless scope for the exercise of the highest efforts of imagination and invention. Lagrange . . . expressed emphatically his belief in the importance to the mathematician of the faculty of observation; Gauss called mathematics the science of the eye . . . I could tell a story of almost romantic interest about my own latest researches in a field where geometry, algebra, and the theory of numbers melt in a surprising manner into one another, . . . which would very strikingly illustrate how much observation, divination, induction, experimental trial, and verification, causation too, . . . have to do with the work of the mathematician." Sylvester proved conclusively that the mathematician engaged in original research does exercise powers of internal observation, induction, experimentation, and even causation. The vital question arises, are these powers exercised by the pupil in the classroom? That depends. If a text-book in geometry is

[1] See page 283, last footnote.

memorized, as used to be the case in some schools, then doubtless Huxley was right. But when attention is paid to original problems, to the use of measuring instruments, to modes of proving theorems different from those in the book, then the pupil exercises the same faculties as does the advanced mathematician engaged in research.

Perhaps the latest prominent attacks on mathematics are by David Snedden, Commissioner of Education in Massachusetts, in an address delivered in 1915 before the National Education Association, and by Abraham Flexner in an article, called "A Modern School."[1] Placing great faith in the methods of experimental pedagogy that are advocated in schools of education of our universities at the present time, these writers are attacking the mind-training value of mathematics and are insisting that, in view of the results obtained from experiments, the burden of proof to show that algebra and geometry should remain in the high school curriculum rests upon those who advocate its continuance and not upon those who would eliminate them. Says Flexner: "Modern education will include nothing simply because tradition recommends it, . . . it includes nothing for which an affirmative cannot now be made out. As has already been intimated, this method of approach would probably result in greatly reducing the time allowed to mathematics, and decidedly changing the form of what is still retained. If, for example, only so much arithmetic is taught as people actually have occasion to use, the subject will sink to modest proportions. . . . The same policy may be employed in dealing with algebra and geometry. What is taught, when it is taught, and how it is taught will in that event depend altogether on what is needed,

[1] *Review of Reviews*, Vol. 53, 1916, pp. 465–474. See also A. Flexner's "Parents and Schools," in the *Atlantic Monthly*, July, 1916, pp. 25–33.

when it is needed, and the form in which it is needed." It will be instructive to watch the outcome of this discussion. Certain it is that the teaching of mathematics will require constant modifications in order to keep pace with the ever-changing needs of modern life.

INDEX

Abacists, 115, 118, 119, 188.
Abacus, 11, 14, 17, 26–28, 37–41, 106, 112, 114, 117–119, 186.
Abel, 227, 238.
Absolute geometry, 273. *See* Non-Euclidean geometry.
Abû Dscha 'far Alchâzin, 110.
Abû Ja'ḳûb Isḥâḳ ibn Ḥunain, 127.
Abû'l Dschûd, 110.
Abul Gud. *See* Abû'l Dschûd.
Abû'l Wafâ, 104, 107, 128, 130, 248.
Achilles and tortoise, paradox of, 59.
Acre, 178, 179.
Adalbold, 132.
Adams, D., 218, 219, 222.
Addition, 26, 37, 38, 96, 101, 105, 110, 115, 145. *See* Computation.
Adelhard of Bath. *See* Athelard of Bath.
Aeneas, 172.
Agrimensores, 89–92.
Ahmes, 19–25, 28, 34, 43–45, 47, 82, 120, 222.
A. I. G. T., 68, 69, 206, 208, 213, 263, 268, 279, 284–286, 288.
Akhmim papyrus, 25, 82.
Albannâ, ibn, 106, 111, 150.
Al Battânî, 118, 130.
Albategnius. *See* Al Battânî.
Albîrûnî, 15, 104.
Alchwarizmî, 104–109, 118, 128, 129, 133, 206.
Alcuin, 112–114, 119, 131, 220.
Aldschebr walmuḳâbala, 107, 108.
Alexandrian School, First, 64–82; Second, 82–89.
Algebra, in Egypt, 23–25; in Greece, 26, 33–37; in Rome, 42; in India, 93–95, 101–103; in Arabia, 104, 106–111; Middle Ages, 118, 120, 122, 133; Modern Times, 139, 141, 156, 183, 206, 209, 224–245, 283, 288; origin of word, 107. *See* Notation.
Algorism, 119; origin of word, 105.
Algoristic school, 118, 119, 188.
Algorithm. *See* Algorism.
Al Hovarezmi. *See* Alchwarizmî.
Aliquot parts, 23, 197, 218.
Alḳalṣâdî, 110, 150.
Al Karchî, 106, 110.
Alligation, 110, 185, 198.
Allman, 47, 48, 50, 51, 53, 56, 57, 59, 62, 63, 66.
Al Mâhânî, 110.
Al Mamûn, 126, 128. Almansur, 129.
Alnager, 176.
Alsted, 203.
Amicable numbers, 29.
Amyclas, 63.
Analysis, in ancient geometry, 61, 62; in arithmetic, 196, 197; algebraic, 249.
Anaxagoras, 48, 49, 53.
Anaximander, 48.
Anaximenes, 48.
Angle, trisection of, 34, 55, 134, 135, 226.
Anharmonic ratio, 258.
Annuities, 198.
Anthology, Palatine, 33, 34, 113.
Anti-parallel, 262.
Antiphon, 57–59.
Apian, 141.
Apices of Boethius, 12, 14–16, 112, 114, 115, 118.
Apollonian Problem, 79.

Apollonius, 65, 75, 78–80, 86, 87, 104, 127, 247, 250, 255.
Apothecaries' weight, 173.
Appuleius, 32.
Arabic notation. *See* Hindu notation.
Arabs, 10, 13–18, 24, 65, 84, 95, 97, 103–111, 116, 118, 119, 124–132, 134, 137, 147, 150, 188, 224, 238.
Archimedes, 28, 33, 63–65, 73, 75–79, 86, 127, 134, 158, 242, 250, 254.
Archytas, 53, 60, 62, 63.
Arenarius, 29.
Argand, 243.
Aristotle, 29, 47, 57, 61, 76, 132, 275.
Arithmetic, in Egypt, 19–26, 82; in Greece, 26–37; in Rome, 37–42; in India, 93–101; in Arabia, 103–111; in Middle Ages, 111–122; in Modern Times, 139–221, 145–168; in England, 20, 168, 179–211; Reforms in teaching, 211–219; in the United States, 215–223. *See* Computation, Hindu notation, Notation, Number-systems.
Arithmetical triangle, 238.
Arneth, 125.
Arnold, M., 207.
Arnold, T., 207.
Artificial numbers. *See* Logarithms.
Aryabhatta, 12, 94, 99, 100, 102, 123.
Assumption, tentative. *See* False position.
Athelard of Bath, 104, 118, 132, 133, 136.
Athenaeus, 63.
Atkinson, 275.
Attic signs, 7.
Austrian method, of subtraction, 213; of division, 214, 215.
Avoirdupois weight, 173, 174, 196, 198.
Axioms, 46, 61, 67, 69–71, 89, 90, 275, 276, 279, 281. *See* Postulates, Parallel-postulate.

Babylonians, 1, 5, 6, 8, 9, 10, 20, 23, 30, 43, 44, 49, 79, 84, 88, 123, 175.
Bache, 215.
Bachet de Méziriac, 220, 222.
Bachmann, 265.
Backer rule of three, 198.
Bacon, R., 137.
Baillet, 25.
Baker, 187, 193.
Bakhshâlî arithmetic, 94, 98, 99.
Ball, W. W. R., 16, 133, 180, 184, 206 242, 281.
Baltzer, R., 273, 280.
Bamberg arithmetic, 16, 140, 180.
Barley-corn, 175, 177.
Barrême, 192.
Barter, 198.
Base, logarithmic, 160.
Basedow, 197.
Battaglini, 278.
Bayle, 246.
Bede, 112, 119.
Behâ Eddîn, 128.
Bekker, 57.
Benese, R. de, 178.
Berkeley, 144.
Bernelinus, 114, 115.
Bernoulli, James, 237.
Bernoulli, John, 200.
Bevan, B., 259.
Beyer, 153.
Bézout, 267, 276.
Bhaskara, 50, 95, 99, 101, 102, 103, 123, 205.
Billingsley, 246, 281.
Billion, 144.
Binomial surds, 103.
Binomial theorem, 209, 229, 237, 238; not inscribed on Newton's tomb, 239.
Biot, 257.
Biquadratic equations, 226.
Bitonto, 267.
Böckler, 154.
Boethius, 12, 14, 15, 16, 32, 41, 91, 92, 111, 112, 114, 115, 117, 118, 131, 132, 133, 136, 142, 147, 188.
Bolyai, J., 70, 71, 86, 127, 268, 269, 273, 274, 275, 279.
Bolyai, W., 273.
Bombelli, 226, 227.
Boncompagni, 105, 142.
Bonnycastle, 244, 267.
Bookkeeping, 121, 168, 188, 206.
Borelli, 268.
Boscovich, 266.
Bouelles. *See* Bouvelles.
Bourdon, 208.

Bouvelles, 247.
Bovillus, 247.
Brackett, 208.
Bradwardine, 136, 137, 247, 250.
Brahe, Tycho, 250.
Brahmagupta, 94, 95, 100, 101, 102, 129, 196, 205.
Bretschneider, 47, 50, 56, 62, 123.
Brewer, 172.
Brewster, 239, 277, 283.
Brianchon, 248, 252, 253, 257, 260.
Brianchon point, 257.
Brianchon's theorem, 257.
Bridge, 244.
Bridges, 188.
Briggian logarithms, 159, 162, 165, 199.
Briggs, 161–163, 167, 179, 238.
Briot and Bouquet, 227.
Brocard, H., 259, 261.
Brocard circle, 261, 262.
Brocard points and angles, 261.
Brocard's first and second triangle, 261.
Brockhaus, 159.
Brocki, J. *See* Broscius.
Brown, 200.
Broscius, 247.
Brute, 172.
Bryson of Heraclea, 58, 59.
Buckley, 150, 187, 188, 189.
Buddha, 29.
Buée, 243.
Burgess, 12.
Bürgi, 153, 166.
Burkhardt, 226.
Bushel, 172.
Butterworth, 260.
Byllion, 144.
Byrgius. *See* Bürgi.

Cæsar, Julius, 90, 91.
Cajori, 26, 29, 159, 237, 238, 304.
Campano. *See* Campanus.
Campanus, 134, 135, 136, 245, 247.
Campbell, T., 137.
Cantor, G., 70.
Cantor, M., 5, 6, 7, 8, 10, 12, 14, 16, 19–26, 31, 34, 38, 43–45, 47, 50, 56, 59, 60, 67, 76, 78, 80, 81, 83–85, 87, 90, 94, 98–102, 105, 110–113, 118, 120, 122, 127, 129, 130, 132, 133, 134, 136, 137, 141–145, 150, 181, 184, 194, 222, 227, 231, 232, 235, 238, 245, 247–249, 250, 252, 282.
Capella, 91, 111, 131.
Cardan, 150, 151, 222, 224, 225, 227 228, 229, 230, 232, 234, 240.
Cardano. *See* Cardan.
Carnot, L., 83, 257, 369.
Casey, 259, 288.
Cassiodorius, 91, 92, 111, 112.
Castellucio, 146.
Casting out the 9's, 98, 106, 149, 184, 185.
Cataldi, 150, 187.
Cattle-problem, 33.
Cauchy, 244.
Cavalieri, 209.
Caxton, 16, 180.
Cayley, 285.
Ceva, 83, 256.
Ceva's theorem, 256.
Chain, 179.
Chain-rule, 196, 197.
Champollion, 6.
Charlemagne, 168, 220.
Charles XII., 2, 205.
Chasles, 74, 83, 87, 89, 248, 252, 255, 256, 258, 259.
Chauvenet, 278.
Chebîchev, 31.
Chinese, 94, 123, 240.
Chuquet, 141, 142, 144, 222.
Cicero, 75.
Cipher, 152; origin of word, 119.
Circle, 48, 52, 53, 54, 57, 58, 59, 73, 75, 78, 79, 87, 250, 255, 265, 276; division of, 10, 43, 73, 74, 79, 84, 124, 134, 265. *See* Squaring of the circle.
Circle-squaring. *See* Squaring of the circle.
Circulating decimals, 200.
Clairaut, 269, 276, 277.
Clausen, Th., 242.
Clavius, 71, 137, 176, 266.
Clifford, W. K., 71, 258, 275, 287.
Cloff, 198.
Cocker, 31, 173, 184, 190, 191, 192–194, 196, 199, 202, 203, 210, 212, 217.
Colburn, W., 218, 219.
Colebrook, 95.

Colla, 224.
Combinations, doctrine of, 237.
Commercial School in England, 179–204.
Common logarithms, 159, 162, 165, 199.
Common notions, 70.
Compasses, single opening of, 128, 248, 264, 265. *See* Ruler and compasses.
Complementary division, 116, 117, 147; —multiplication, 147.
Complex numbers, 244. *See* Imaginaries.
Compound interest, 151. *See* Interest.
Computation, modes of, in Ahmes, 21–24; in Greece, 27, 28, 32, 37–42; in India, 95–100; in Arabia, 104 106; Middle Ages, 111, 112, 114–122; Modern Times, 145–167, 186, 187, 206; in England, 205, 206, 241; in the United States, 241.
Computus, 112.
Comte, v.
Conant, 1, 3, 4, 5.
Conic sections, 63, 78, 254, 255.
Conjoined proportion. *See* Chain-rule.
Continued fractions, 150, 200.
Continuity, law of, in algebra, 237, 287; in geometry, 237, 252, 268, 287.
Cooley, 267.
Cosine, 124, 162.
Cossic art, 228. *See* Algebra.
Cotangent, 130, 137, 163.
Cotes, 208, 256.
Counters, 115, 118, 185–187.
Crelle, 227, 238, 260, 261.
Cremona, 248, 278, 279.
Çridhara, 95, 102.
Cross-ratio, 255, 258.
Ctesibius, 80.
Cube, duplication of, 53, 54, 56, 226.
Cube root, 101, 106, 150, 228, 241.
Cubic equations, 110, 120, 129, 224–226, 227, 229, 230, 240.
Cubical numbers, 32.
Cubit, 175.
Cunn, 55, 200.
Curtz, 134, 222.
Cyzicenus, 63.
Daboll, 218.
Damascius, 67, 88, 127.
Dase, 166, 242.
Data of Euclid, 74, 133, 246.
Davies, 189.
Davies, C., 277.
Davies, F. S., 260.
Da Vinci, Leonardo, 141, 248, 249.
De Benese, 178.
Decimal fractions, 150–155, 166, 187 199, 200, 214.
Decimal point, 153, 154, 194, 195.
Decimal scale, 1, 2, 3, 4, 5, 8, 11, 40, 85, 152, 163, 165, 166, 178.
Dedekind, 72.
Dee, John, 246.
Defective numbers, 29, 113.
Degrees, 10, 179.
De Jonquières, 265.
De Lagny, 241, 242.
De Lahire, 252, 253.
De la Hire. *See* De Lahire.
Delbœuf, 269.
Del Ferro. *See* Ferro.
Delian Problem, 55. *See* Duplication of the cube.
Democritus, 45, 46.
De Morgan, 55, 64, 65, 68, 72, 122, 141, 151, 154, 164, 175, 178, 181, 187–190, 199, 200, 205, 208, 210, 213, 238, 240, 241, 244, 246, 268, 283, 284, 286.
Demotic writing, 7.
Desargues, 237, 252, 253, 255, 257, 269.
Desargues' Theorem, 253.
Descartes, 233, 234, 235, 236, 238, 243, 252, 254, 256, 265.
Descartes' Rule of Signs, 236.
Devanagari-numerals, 14, 16, 17.
Dilworth, 160, 174, 191, 194, 195, 198, 201, 202, 203, 217, 218, 219, 220.
Dinostratus, 63.
Diogenes Laertius, 28, 47, 61.
Diophantine analysis, 37, 101.
Diophantus, 26, 34–37, 82, 87, 101, 102, 104, 108, 109, 127.
Direction, concept of, in geometry, 268, 280, 289.
Dirichlet, 31.
Divisio aurea, 117.
Divisio ferrea, 117.
Division, 37, 39, 98, 101, 105, 114, 115–

118, 140, 148, 149, 150, 151, 154, 156, 183, 189, 192, 213, 214.
Dixon, E. T., 68, 268.
Dodecaedron, 51. *See* Regular solids.
Dodgson, 267, 268, 270.
Dodson, 195, 199.
Dollar mark, 223.
Double position, 106, 196, 198, 203. *See* False position.
Dschâbir ibn Aflaḥ, 107, 129, 130.
Duality, 258.
Dücange, 143.
Duhamel, 77, 278.
Duodecimal fractions, 38–41, 119.
Duodecimal scale, 2, 3, 112, 117.
Dupin, 257.
Duplication of the cube, 53, 54, 56, 226.
Durège, 266.
Dürer, 221, 248, 249, 266.

Easter, computation of time of, 40, 112. *See* Computus.
Egyptians, 1, 6, 7, 19–26, 27, 44–46, 49, 52, 55, 76, 88–90, 99, 122, 131, 170, 175.
Eisenlohr, 19.
Elefuga, 137.
Elements of Euclid. *See* Euclid's Elements.
Emmerich, 259, 262.
Eneström, 141.
Engel and Stäckel, 68, 70, 72, 267, 269–272, 274.
Epicureans, 74.
Equality, symbol for, 184, 195.
Equations, linear, 23, 33, 37, 107, 108; quadratic, 36, 101, 102, 108, 110, 230; cubic, 110, 120, 129, 224–227, 229, 240; simultaneous, 35; higher, 226, 227, 230, 239; numerical, 240; theory of, 230–233, 236, 243.
Eratosthenes, 31, 56, 65.
Eton, 204, 205, 209.
Euclid, 31, 35, 46, 47, 50, 52, 60, 62, 63, 64–75, 76, 77, 79, 80, 83, 85, 86, 104, 134, 135, 272, 274, 276, 277, 278.
Euclid of Megara, 65, 246.
Euclid's Elements, 30, 65–75, 79, 81, 84, 87, 88, 91, 103, 126, 127, 133, 136–138, 206, 207, 212, 238, 245, 246, 254, 270, 275–289; editions of the Elements, 245, 246, 266, 282, 288.
Euclid and his modern rivals, 267, 268, 275–289.
Eudemian Summary, 47–49, 61, 63, 64.
Eudemus, 47, 52.
Eudoxus, 46, 60, 62, 63, 64, 66.
Euler, 36, 157, 195, 200, 238, 243, 259, 265.
Eutocius, 26.
Evolution. *See* Square root, Cube root, Roots.
Excessive numbers, 29, 111.
Exchange, 121, 198.
Exchequer, 174, 186, 215.
Exhaustion, method of, 59, 60, 62, 63, 66.
Exhaustion, process of, 57, 59.
Exponents, 122, 152, 183, 234–238.

Fallacies, 74.
False position, 24, 36, 99, 106, 141, 184, 185, 196, 198, 240.
Farrar, J., 243, 277.
Farthing, derivation of word, 170.
Fellowship, 121, 185.
Fenning, 191, 217.
Fermat, 209, 252.
Ferrari, 225, 226, 228.
Ferreus. *See* Ferro.
Ferro, 224, 225.
Feuerbach, 260.
Feuerbach circle, 259, 260. *See* Nine-point circle.
Fibonacci. *See* Leonardo of Pisa.
Figure of the Bride, 128.
Finæus, 147, 150, 151.
Finger-reckoning, 1, 2, 17, 27, 40, 112.
Fiore, Antonio Maria. *See* Floridus.
Fisher, George, 217, 192.
Fisher, G. E., 266.
Flamsteed, 242.
Florido. *See* Floridus.
Floridus, 224, 225.
Foot, 175, 176.
Fourier, 240.
Fractions, 20–24, 38–41, 82, 98, 106, 112, 114, 122, 150–155, 181–183, 185, 201, 202, 207. *See* Decimal fractions.
Français, 243.

Friedlein, 8, 14, 26, 32, 37–40, 64, 116, 117, 147.
Frisius. *See* Gemma-Frisius.
Froebel, v.
Frontinus, 91, 92.
Furlong, 177.

Galileo, 155, 250.
Galley method of division, 148, 149. *See* Scratch method.
Gallon, 172, 176, 177.
Garnett, 16.
Gauss, 31, 36, 74, 244, 247, 273, 274, 279, 283.
Geber, 107. *See* Dschâbir ibn Aflaḥ.
Gellibrand, 163.
Geminus, 52, 84.
Gemma-Frisius, 203.
Geometry, in Egypt, 44–46; in Babylonia, 43; in Greece, 30, 46–89; in Rome, 89–92; in India, 93, 122–124; in Arabia, 125–129; Middle Ages, 131–138; in England, 135, 205, 206, 208, 245, 246, 259, 260, 263, 267, 281–289; Modern Times, 230, 245–289; modern synthetic, 252–259; editions of Euclid, 245–247, 266, 282, 288; modern geometry of triangle, 259–263; non-Euclidean geometry, 266–275; text-books, 275–289.
Gerard of Cremona, 118, 133, 134.
Gerbert, 106, 114–119, 131, 132, 147.
Gergonne, 253, 258, 260.
Gerhardt, 118, 167, 194.
Gibbs, J. W., 245.
Girard, A., 153, 230, 231, 233, 234, 247.
Glaisher, 163.
Golden rule, 185, 189, 193, 201. *See* Rule of Three.
Golden section, 63.
Gow, 19, 20, 26, 29, 31–33, 36, 44, 47, 50, 55–57, 60–62, 65, 67, 76, 78, 81, 85, 133, 136.
Graap, 50.
Grain of barley, 170; of wheat, 172, 173, 174.
Grammateus, 141.
Gramme, 171.
Grassmann, 245.
Graves, 55, 241.
Grebe point, 262.
Greeks, 1, 4, 7, 8, 10, 20, 26–37, 46–89, 91, 94, 102, 123, 126, 171, 175, 248, 263.
Greenwood, I., 217.
Greenwood, S., 217.
Gregory, D., 245.
Gregory, D. F., 244.
Gregory, James, 242.
Groat, 178; derivation of word, 170.
Gromatici, 89–92.
Grube, 212.
Grynæus, 245.
Gubar-numerals, 13, 14, 16, 17, 110.
Guinea, derivation of word, 170.
Gunter, 163, 179, 199.
Günther, 8, 13, 17, 117, 132, 133, 135, 137, 159, 194, 220, 221, 247, 248, 250, 266.

Hachette, 257.
Haddschâdsch ibn Jûsuf ibn Maṭar, 126.
Halifax, John, 122.
Hallam, 228.
Halliwell, 122, 133, 135, 181.
Halsted, 67, 68, 70, 71, 74, 77, 246, 266, 268, 270–275.
Hamilton, W. R., 55, 245. W., 304.
Hankel, 5, 26, 36, 40, 47, 50, 51, 52, 56, 59, 60–62, 70, 72, 89, 90, 96, 97, 100, 103, 105–107, 116, 121, 129, 132, 133, 137, 143, 230, 240, 244, 287, 288.
Hardy, A. S., 243.
Harpedonaptæ, 45.
Harriot, 157, 231, 232, 234.
Harrow, 204, 207.
Hârûn ar-Raschîd, 126.
Hassler, 215.
Hastings, 205.
Hatton, 156, 195, 199, 202.
Hawkins, 190, 193.
Hayward, R. B., 241.
Heath, 33–36.
Hebrews, 44, 49, 175.
Heckenberg, 144.
Hegesippus, 222.
Heiberg, 30, 67, 69, 75, 84.
Heiberg and Menge, 70, 71, 246.
Helicon, 63.
Helmholtz, 275.
Hendricks, 227.
Henrici and Treutlein, 280.

Henry I., 175, 186.
Henry VII., 167.
Henry VIII., 169, 172, 246.
Heppel, v, 206.
Hermite, 227.
Hermotimus, 63.
Herodianic signs, 7.
Herodianus, 7.
Herodotus, 27.
Heron of Alexandria, 79–82, 90, 102, 113, 127, 250.
Heron the Elder. *See* Heron of Alexandria.
Heron the Younger, 80.
Heronic formula, 80, 90, 123, 128, 134.
Heteromecic numbers, 29.
Hexagramma mysticum, 255.
Hieratic symbols, 7.
Hieroglyphics, 6, 45.
Hill, J., 199, 201, 203.
Hill, T., 219.
Hindu notation, 11, 13–18, 145, 154, 155.
Hindu numerals, 12–18, 107, 119, 121, 179, 180, 181, 187.
Hindu proof, 106. *See* Casting out the 9's.
Hindus, 8, 9, 11, 12, 15, 24, 82, 88, 93–103, 113, 116, 119, 122–124, 128, 136, 143, 145, 147, 149, 150, 170, 215, 221, 227, 233, 238, 250.
Hipparchus, 79, 84.
Hippasus, 51, 52.
Hippias, 55, 56.
Hippocrates of Chios, 56, 57, 59, 60, 61, 65.
Hirst, 279, 284, 286.
Hodder, 191, 217.
Hoefer, 133.
Hoernle, 94.
Hoffmann, 50, 128.
Hoffmann's Zeitschrift, 280.
Holywood, 122.
Homological figures, 253.
Homology, law of, 288.
Hook, 208.
Horace, 40.
Horner, 226, 240.
Horner's method, 240, 241. *See* Horner.
Hoüel, 278.
Hughes, T., 232.
Hultsch, 65, 81.
Hume, 167, 168.
Hunain, 127.
Ḥunain ibn Isḥâḳ, 127.
Hutt, 264.
Hutton, 189, 243, 244, 264.
Huxley, 283, 306.
Hypatia, 87.
Hyperbolic logarithms, 166, 167. *See* Natural logarithms.
Hypothetical constructions, 73, 74, 277.
Hypsicles, 28, 31, 67, 79, 127.

Iamblichus, 27, 30, 33, 82, 108.
Ibn Albannâ, 106, 111, 150.
Imaginary geometry, 272. *See* Non-Euclidean geometry.
Imaginary quantities, 227, 231–233, 236, 243–245.
Inch (uncia), 40, 171, 175.
Incommensurables, 63, 66, 72, 266, 278, 280, 285–288. *See* Irrational quantities.
India. *See* Hindus.
Indices. *See* Exponents.
Infinite, 59, 70, 136, 250, 252, 278; symbol for, 237.
Infinite series. *See* Series.
Infinitesimal, 136.
Insurance, 200.
Interest, 100, 121, 151, 167, 168, 198.
Inversion, 99.
Involution of points, 87, 252, 253.
Ionic School, 47–49.
Irrational quantities, 29, 51, 67, 72, 76, 101, 102, 103, 108, 287. *See* Incommensurables.
Isidorus, 88.
Isidorus of Carthage, 92, 111.
Isoperimetrical figures, 79, 136.
Italian method of division, 213.

Jakobi, C. F. A., 260, 261.
Joachim, 251. *See* Rhæticus.
John of Palermo, 120.
John of Seville, 118, 119, 150.
Johnson, 3.
Jones, 283.

Jones, W., 200.
Jonquières, de, 265.
Jordanus Nemorarius, 134, 135, 142.
Joseph Sapiens, 117.
Joseph's Play, 222.
Josephus, 222.

Kambly, 280.
Kant, 277,
Kästner, 127, 181, 207, 279.
Kelly, 168, 169, 170, 174, 196.
Kempe, 264.
Kepler, 136, 155, 166, 167, 237, 247, 249, 252, 269.
Kersey, 153, 194, 196, 199, 201, 203, 220, 221.
Kettensatz, 196. *See* Chain-rule.
Kingsley, 87.
Kircher, 247.
Klein, 70, 74, 249, 264, 265.
Koppe, K., 280.
Krause, 247.
Kronecker, 227.
Kruse, 280.

Lacroix, S. F., 268, 276.
Laertius, Diogenes, 28, 47, 61.
Lagny, de, 241, 242.
Lagrange, 36, 214, 240, 270, 283.
Lahire, de, 252, 253.
Lambert, 242, 248, 269, 274.
Lange, J., 259.
Langley, E. M., 213, 241.
Langley and Phillips, 288.
Laplace, 155, 269.
La Roche, 142, 144.
Larousse, 276.
Lefère, 142.
Legendre, 31, 243, 269, 270, 271, 272, 276, 277, 278, 283, 286.
Leibniz, 195, 254, 262.
Lemma of Menelaus, 83.
Lemoine, E., 259, 262.
Lemoine circles, 262.
Lemoine point, 262.
Leodamas, 63.
Leon, 63, 65.
Leonardo da Vinci, 141, 248, 249.
Leonardo of Pisa, 24, 76, 119–121, 134, 135, 139, 141, 150, 196, 222.
Leudesdorf, 248.
Leyburn, 203.
Limits, 77, 78, 250, 278, 287.
Lindemann, 243.
Linkages, 263, 264.
Lionardo da Vinci. *See* Vinci.
Lippich, 265.
Littré, E., 144.
Lobatchewsky, 86, 127, 266, 272–275, 279.
Local value, principle of, 8, 9, 10, 11, 12, 15, 42, 95, 105, 118.
Locke, 144.
Logarithmic series, 167.
Logarithmic tables, 160–166.
Logarithms, 155–167, 187, 199, 229.
Lorenz, 269.
Loria, 25, 47, 48, 50, 56, 59, 65, 67, 76, 277–280, 283.
Lowe, 189.
Lübsen, 279.
Luca Paciuolo. *See* Pacioli.
Lucas di Burgo. *See* Pacioli.
Lucian, 31.
Ludolph van Ceulen, 242.
Ludolph's number, 242.
Ludus Joseph, 222.
Lune, 57.
Lyte, 152.

Macdonald, 156, 160, 194, 206.
Machin, 242.
Mackay, 259, 262, 288.
Mackay circles, 263.
Maclaurin, 208, 244, 256.
Magic squares, 221.
Mahavira, 95.
Malcolm, 200.
Malfatti, 226.
Marcellus, 75, 78.
Marie, 80, 133, 199, 252, 264
Marsh, 200.
Martin, B., 194, 199.
Martin, H., 218.
Mascheroni, 264.
Mathematical recreations, 24, 113, 219–223.
Matthiessen, 23, 227, 234.
Maudith, 137.
Maxwell, 77.

McLellan and Dewey, 212.
Mean proportionals, 56.
Measurer by the ell, 176.
Medians of a triangle, 134.
Meister, 247.
Mellis, 188.
Menæchmus, 63.
Menelaus, 82, 85, 129, 133; lemma of, 83.
Menge. *See* Heiberg and Menge.
Mental arithmetic, 204, 219.
Mercator, N., 167.
Merchant Taylors' School, 204, 206, 207.
Methods. *See* Teaching of Mathematics.
Metric System, 152, 171, 178, 179, 215.
Méziriac, 220, 222.
Million, 143, 144.
Minus, symbol for, 141, 184, 228. *See* Subtraction.
Minutes, 10.
Möbius, 247, 258.
Moivre, de, 208.
Moneyer's pound, 168. *See* Tower pound.
Monge, 257.
Morgan, de. *See* De Morgan.
Moschopulus, 221.
Motion of translation, 70, 135, 157; of reversion, 70; circular and rectilinear, 263, 264.
Muḥammed ibn Mûsâ Alchwarizmî, 104–109, 118, 128, 129, 133, 206.
Müller, F., 107.
Müller, H., 268, 280.
Müller, John, 250. *See* Regiomontanus.
Multiplication, 26, 96–98, 101, 105, 110, 115, 140, 145–147, 156, 189, 194, 195, 213, 214, 234; of fractions, 21, 22, 181, 182.
Multiplication table, 32, 39, 40, 115, 117, 147, 148, 181.
Murray, 173.
Mûsâ ibn Schâkir, 128.
Mystic hexagon, 255.

Naperian logarithms, 159.
Napier, M., 154.
Napier, J., 149, 153, 155–167, 194, 206, 229, 251.
Napoleon I., 69, 257, 264.
Nasîr Eddîn, 127, 128, 130, 131, 270.
Natural logarithms, 159, 160, 164, 166.
Naucrates, 65.
Negative quantities, 35, 88, 101, 102, 107, 227, 230–233, 243, 244, 258.
Nemorarius, 134, 135, 142. *See* Jordanus Nemorarius.
Neocleides, 63.
Neper. *See* Napier.
Nesselmann, 30, 31, 37, 72, 108, 109.
Neuberg circles, 263.
Newcomb, 68, 288.
Newton, 55, 77, 167, 170, 194, 208, 209, 226, 237–240, 243, 255, 273.
Nicolo Fontana. *See* Tartaglia.
Nicomachus, 26, 31–33, 41, 91, 92.
Niedermüller, 214.
Nine-point circle, 259, 260.
Nixon, 285, 286, 288.
Non-Euclidean geometry, 266–275.
Norfolk, 180, 181.
Norton, R., 152.
Notation, of numbers (*see* Numerals, Local value); of fractions, 21, 26, 98, 106, 181; of decimal fractions, 152–154; in arithmetic and algebra, 23, 30, 35, 36, 98, 101, 108–111, 122, 184, 194, 195, 228, 229, 234, 235, 237, 239.
Number, concept of, 29, 35, 232, 233. *See* Negative quantities, Imaginaries, Irrationals, Incommensurables.
Numbers, amicable, 29; cubical, 32; defective, 29, 113; excessive, 29, 111; heteromecic, 29; prime, 30, 31, 74; perfect, 29, 111, 113; polygonal, 31, 32; triangular, 29.
Number-systems, 1–18.
Numerals, 6, 7, 8, 12–16, 32, 38, 107, 119, 121, 180.
Numeration, 1–5, 8, 12–18, 142–145.

Observation in mathematics, 74, 77, 78, 151, 155, 283, 288.
Octonary numeration, 3.
Œnopides, 46.
Oinopides. *See* Œnopides.

Ohm, M., 244.
Oldenburg, 237.
'Omar Alchaijâmî, 110.
Oresme, 122, 234.
Otho, V., 251.
Ottaiano, 87.
Oughtred, 149, 153, 154, 187, 194, 195, 203, 234.
Ounce (uncia), 40, 112, 171, 172, 173.
Oxford, university of, 137, 138.

π, 44, 49, 73, 76, 123, 128, 145, 241–243. *See* Squaring of the circle.
Pacciuolus. *See* Pacioli.
Pacioli, 105, 139, 141–143, 145, 146–148, 181, 182, 188, 196, 208, 214, 222, 232, 238, 245.
Paciuolo. *See* Pacioli.
Padmanabha, 95.
Palatine anthology, 33, 34.
Palgrave, 186, 256.
Palms, 175.
Paolis, R. de, 279.
Paper folding, 265.
Pappus, 65, 79, 82, 86–89, 248, 255.
Parallel lines, 43, 52, 85, 86, 127, 252, 266–275, 280, 282.
Parallel-postulate, 70, 71, 85, 86, 127, 266–275, 279, 282, 283, 289; stated, 71.
Parley, Peter, 218.
Pascal, 238, 252–255.
Pascal's theorem, 255, 257.
Pathway of Knowledge, 173, 174, 188, 189.
Peacham, 205, 210.
Peacock, 2, 121, 143, 145–150, 153, 181, 182, 186, 187, 193, 198, 220, 244.
Peaucellier, 263, 264.
Pedagogics. *See* Teaching.
Peirce, B., 245.
Peirce, C. S., 67, 70.
Peletier, 230, 266.
Pentagram-star, 136.
Pepys, 192.
Perch, 178.
Perfect numbers, 29, 111, 113.
Perier, Madam, 254.
Perry, J., 291.
Pestalozzi, v, 197, 203, 211, 212, 219.
Petrus, 136.
Peuerbach. *See* Purbach.
Peurbach. *See* Purbach.
Peyrard, 71.
Philippus of Mende, 63.
Philolaus, 29, 53.
Phœnicians, 208.
Picton, 180.
Pike, 214, 216–218.
Pitiscus, 203, 251.
Planudes, 16, 95, 213.
Plato, 26, 29, 46, 50, 53, 54, 56, 60–63, 65, 114, 208.
Plato of Tivoli, 118, 124, 130, 134.
Plato Tiburtinus. *See* Plato of Tivoli.
Platonic figures, 65, 73, 78.
Platonic school, 53, 60–63.
Playfair, 270, 271, 282.
Plus, symbol for, 141, 184, 228. *See* Addition.
Plutarch, 47.
Poinsot, 247.
Polars, 257, 258.
Polygonal numbers, 31, 32.
Poncelet, 248, 252, 253, 257, 258, 260.
Pons asinorum, 137.
Porisms, 74.
Porphyrius, 82.
Position, principle of. *See* Local value.
Postulates, 46, 54, 61, 67, 69, 70, 71, 73, 89, 134, 266–275, 279, 281.
Pothenot, 251.
Pothenot's problem, 251.
Pott, 4.
Pound, 40, 168–175, 178, 185, 218.
Practice, 22, 189, 196.
Primes, 30, 31, 74.
"Problems for Quickening the Mind," 113, 131, 220.
Proclus, 47, 61, 63, 70, 74, 85, 88, 247.
Progressions, arithmetical, 9, 23, 24, 31, 100, 158, 160, 181, 185, 237; geometrical, 9, 23, 24, 100, 158, 160, 181, 237.
Projection, 258.
Proportion, 30, 31, 47, 53, 57, 65, 67, 72, 73, 110, 196, 197, 202, 234, 277, 282, 285, 286. *See* Rule of Three.
Ptolemæus. *See* Ptolemy.

Ptolemy, 28, 82, 84, 104, 123, 124, 126, 129, 130, 270.
Ptolemy I., 64.
Public schools in England, 204.
Purbach, 140, 148, 149, 213, 250.
Pythagoras, 46, 48–53, 123, 136; theorem of, 45, 49–51, 56, 66, 123, 128, 137.
Pythagorean school, 49–53.
Pythagoreans, 14, 27, 29, 30, 49–53, 66, 136, 250.

Quadratic equations. *See* Equations.
Quadratrix, 55.
Quadrature of the circle. *See* Squaring of circle.
Quadratures, 167, 237.
Quadrivium. *See* Quadruvium.
Quadruvium, 92, 111, 114.
Quick, v, 212.
Quinary scale, 1, 3, 4.
Quotient, 37, 115, 183.

Radian, 251.
Radical sign, 228, 229, 235, 236.
Raleigh, W., 231.
Ralphson, 55.
Ramus, 247, 275.
Ratio, 32, 51, 63, 69, 76, 87, 167, 188, 194, 195, 234.
Raumer, 286.
Reciprocal polars, 258.
Recorde, 118, 147, 149, 183–188, 194, 196, 198, 206, 208, 228, 234, 281.
Reddall, 204, 207.
Reductio ad absurdum, 58, 60, 62.
Rees, A., 166.
Rees, K. F., 196, 197.
Reesischer Satz, 196. *See* Chain-rule.
Regiomontanus, 108, 136, 140, 181, 208, 229, 247, 250, 251.
Regular solids, 52, 62, 65, 67, 73, 78, 128, 136, 249.
Reiff, 238.
Rhæticus, 187, 251.
Rheticus. *See* Rhæticus.
Rhetorical algebras, 108, 120.
Rhind papyrus, 19. *See* Ahmes.
Richards, E. L., 74, 268.
Riemann, 31, 275, 283.
Riese, Adam, 31, 142, 192, 196, 213.
Rigidity of figures, 280.
Rigidity postulate, 71.
Roberval, 209.
Robinson, 216, 267.
Roche. *See* La Roche.
Rodet, 94.
Rods, Napier's, 156, 157.
Rohault, 65.
Roman notation, 4, 8, 11, 121, 180.
Romans, 1, 4, 8, 20, 37–42, 82, 88–92 116, 122, 131, 135, 168, 175.
Romanus, A., 242, 251.
Roods, 178.
Root (*see* Square root, Cube root), 36, 72, 76, 101, 102, 106, 108, 110, 119, 150, 151, 200, 227–233, 235, 236, 241.
Rosenkranz, 208.
Rouché and C. de Comberousse, 278, 287.
Row, S., 265.
Rudolff, 141, 143, 151, 228, 229, 232.
Ruffini, 226.
Rugby, 204, 207.
Rule of Falsehode, 184. *See* False position.
Rule of four quantities, 129.
Rule of six quantities, 83, 129.
Rule of Three, 100, 106, 121, 185, 193–196, 198, 201, 202. *See* Proportion.
Ruler and compasses, 54, 135, 248, 249, 264.
Ruler, graduated, 134.
Rutherford, 242.

Saccheri, 267, 269, 271, 274, 278.
Sacrobosco, 122.
Sadowski, 148, 213, 215.
Salvianus Julianus, 41.
Sand-counter, 29.
Sannia and E. D'Ovidio, 279, 287.
Sanscrit letters, 16.
Saunderson, 244.
Savile, 281, 282.
Schilke, 79.
Schlegel, 280.
Schlussrechnung, 196, 197. *See* Analysis in arithmetic.
Schmid, K.A., 143, 286.
Schmidt, F., 273.

Schotten, H., 230.
Schreiber, H., 141.
Schröter, 265.
Schubert, H., 249.
Schulze, 166.
Schwatt, 266.
Scratch method of division, 148, 149, 192, 215; of multiplication, 105.
Scratch method of square root, 150.
Screw, 55.
Secant, 130, 137, 251.
Seconds, 10.
Seneca, 65.
Serenus, 82–84.
Series, 32, 110, 120, 167, 229, 238, 239, 242, 243. *See* Progression.
Serret, 227.
Servois, 248, 257.
Sexagesimal fractions, 10, 20, 106, 119, 120, 151.
Sexagesimal scale, 6, 9, 10, 28, 43, 79, 84, 85, 124, 163.
Sextus Julius Africanus, 86.
Shanks, 145, 242.
Sharp, A., 241, 242.
Sharpless, 204, 207.
Shelley, 173, 187, 199.
Shillings, 168, 169, 170, 185, 218.
Sieve of Eratosthenes, 31.
Simplicius, 58.
Simpson, T., 243, 244, 267, 284.
Simson, R., 67, 69, 71, 74, 246, 256, 282, 285.
Sine, 85, 124, 129, 130, 159, 162, 163, 251; origin of word, 124, 130.
Singhalesian signs, 12.
Single position, 196. *See* False position.
Slack, Mrs., 192, 217.
Sluggard's rule, 147.
Snellius, 251.
Socrates, 60, 65.
Sohncke, 83.
Solids, regular. *See* Regular solids.
Solon, 7, 28.
Sophists, 26, 53–60, 276.
Sosigenes, 91.
Speidell, E., 165.
Speidell, J., 164, 165.
Spencer, v.
Spenser, 172.
Sphere, 52, 62, 63, 78, 79, 249.
Spherical geometry, 73, 83, 125, 156, 247. *See* Sphere.
Square root, 28, 76, 101, 106, 110, 150, 151, 154, 185.
Squaring of the circle, 54, 55, 57, 58, 200, 226, 249, 271, 282. *See* π.
Stäckel. *See* Engel and Stäckel.
Star-polyædra, 247, 250.
Star-polygons, 136, 247.
Staudt, von, 258, 259, 265.
Steiner, 248, 258, 264.
Stephen, L., 133.
Sterling, 172; origin of word, 170.
Stevin, 122, 149, 151–153, 157, 234, 238.
Stewart, M., 256.
Stifel, 141, 143, 149, 157, 158, 208, 228, 232, 233, 234, 238.
Stobæus, 65.
Stone, 201, 240, 262, 267, 269.
Sturm, 253.
St. Vincent, 167.
Subtraction, 26, 37, 38, 96, 101, 105, 110, 115, 145.
Surds, 150.
Surya-siddhanta, 12.
Surveying, 89, 91, 136, 178.
Suter, 128–130, 133, 137.
Sylvester, 22, 264, 283, 285.
Symbolic algebras, 109.
Symmedian point, 259, 262.
Syncopated algebras, 108.
Synthesis, 62.

Tâbit ibn Ḳurrah, 127, 129.
Tables, logarithmic, 160–166; multiplication, 32, 39, 40, 115, 117, 147, 148, 181; trigonometric, 79, 84, 124, 125, 130, 159, 250, 251.
Tacquet, 66, 188.
Tägert, 70.
Tangent in trigonometry, 130, 137, 250, 251.
Tannery, 47, 56, 81, 128.
Tartaglia, 139, 145, 150, 188, 193, 196, 198, 208, 220, 222, 224, 225, 228, 248.
Taylor, B., 208.
Taylor, C., 252, 257, 264.
Taylor, H. M., 68, 288.
Taylor circles, 263.

Teaching, hints on methods of, v, 17, 25, 41, 74, 78, 114, 138, 151, 155, 183, 185, 189, 197, 198, 211, 212, 233, 236, 238, 239, 277, 281, 288.
Telescope, 155, 232.
Thales, 46–49, 52.
Theætetus, 63, 64, 66.
Theodorus, 60.
Theodosius, 133.
Theon of Alexandria, 28, 69, 79, 82, 87, 91.
Theon of Smyrna, 33, 82.
Theory of numbers, 37, 51, 82, 112.
Theudius, 63, 65.
Thompson, T. P., 279.
Thymaridas, 33, 108.
Timæus of Locri, 60.
Timbs, 204, 205, 207, 210.
Todhunter, 62, 68, 69.
Tonstall, 16, 143, 149, 180–183, 185, 187, 188, 203, 206, 208.
Torporley, 251.
Tower pound, 168, 171, 172, 174.
Transversals, 83, 252, 255, 258.
Treutlein, 13.
Triangular numbers, 29.
Trigonometry, 79, 81, 84, 85, 123–125, 129–131, 136, 139, 155, 156, 159, 165, 230, 243, 250, 251, 270.
Trisection of an angle, 34, 55, 134, 135, 226.
Trivium, 92, 114.
Troy pound, 169, 170, 172, 174; derivation of word, 172.
Tucker, 262.
Tucker circle, 262, 263.
Tycho Brahe, 250.
Tylor, 3, 4, 17, 204.
Tyndale, 183.

Unger, 16, 121, 140, 142, 144, 147, 148, 181, 191, 194, 195, 204, 207, 212, 215.
University of Oxford, 137, 138.
University of Paris, 137.
University of Prague, 137.
Usury, 168.

Vacquant, Ch., 278.
Valentin, 87.
Van Ceulen, 242.
Varignon, 267.
Varro, 91.
Vasiliev, 272.
Venturi, 81.
Veronese, G., 279.
Versed sine, 124.
Vesa, 242.
Victorius, 40.
Vieta, 79, 109, 187, 226, 228–232, 233, 234, 238, 240, 242, 250, 251.
Vigarié, E., 259.
Vigesimal scale, 2, 3, 4.
Villefranche, 142.
Vinci, Leonardo da, 141, 248, 249.
Vlacq, 163, 167.
Von Staudt, 258, 259, 265.

Wafâ. *See* Abû'l Wafâ.
Wagner, U., 140.
Walker, F. A., 169.
Walker, J. J., 157.
Wallace line and point, 259.
Wallis, 123, 128, 149, 153, 164, 200, 208, 209, 236, 237, 269, 274.
Wantzel, 227.
Ward, 200, 209.
Warner, W., 232.
Webster, W., 210.
Weights and measures, 9, 140, 152, 166–179; in United States, 215–217.
Weissenborn, 76, 106, 117, 255.
Wells, 205.
Wessel, C., 243.
Whewell, 281.
Whiston, 66, 239.
Whitehead, 246.
Whitley, 260.
Whitney, 12.
Widmann, 140, 141, 196.
Wiener, H., 265.
Wilder, 194.
Wildermuth, 143, 153, 154, 189, 195, 222.
William the Conqueror, 168, 171.
Williamson, 246.
Wilson, J. M., 268, 283.
Winchester bushel, 176. *See* Bushel
Winchester (school), 204, 206.
Windmill, 137.

Wingate, 153, 173, 187, 194, 195, 196, 199, 201, 202, 203, 220, 221.
Wipper, 50, 128.
Wolf, 194, 266.
Wolfram, 166.
Wöpcke, 14, 15.
Wordsworth, C., 244.
Worpitzky, 280.
Wright, 153, 200, 206.

Xenocrates, 61.

Yancos, 145.
Yard, 175, 176.
Young, 6.

Zambertus, 245.
Zeno, 58, 59.
Zenodorus, 79, 136, 247.
Zero, 10, 11, 12, 14, 15, 42, 95, 99, 115, 118, 119, 143, 153.
Ziwet, 280.
Zschokke, 215.

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